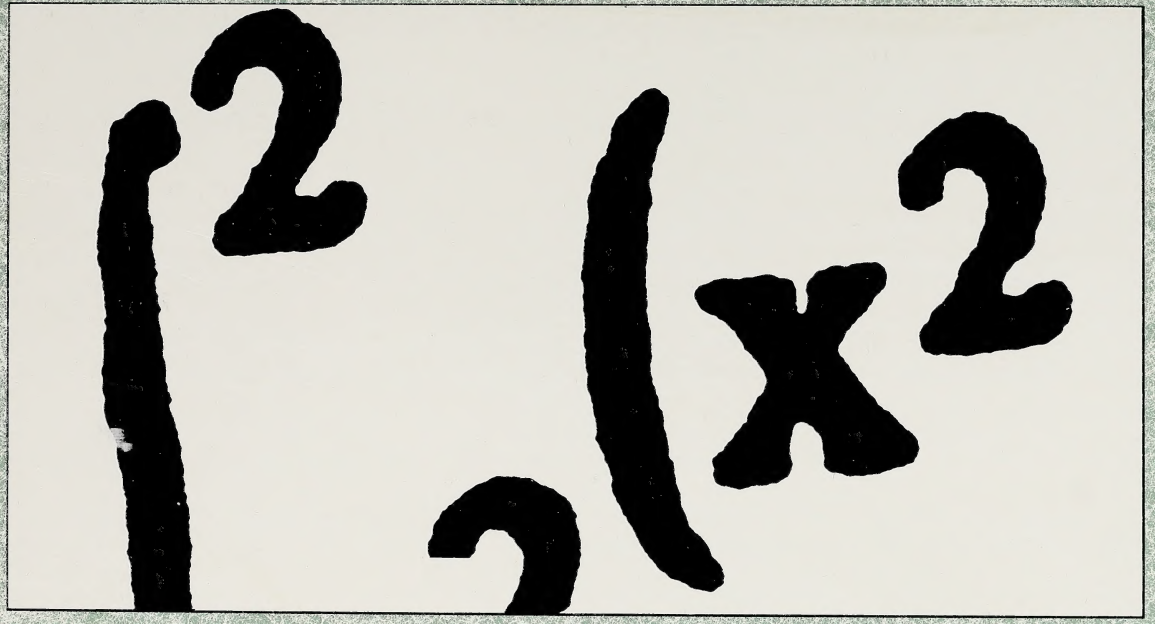
MATHEMATICS 31



Distance Learning

UNIT 4: INTEGRATION

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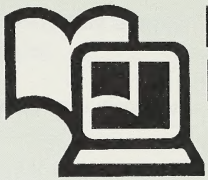
Welcome



You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 31 Student Module Unit 4 Integration Alberta Distance Learning Centre ISBN No. 0-7741-0282-9

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General Information

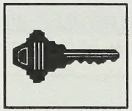
This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.
- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.
- **Extensions** gives you the opportunity to take the topic one step further.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.
- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.

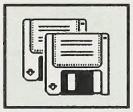


Key Idea
- flagging important ideas

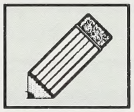
Audiotape
- learning by listening to an audiotape

What You Already Know
- reviewing what you already know

Another View
- exploring different perspectives



Computer Software
- learning by using computer software



Review
- studying previous concepts

Solutions
- correcting the activities

Videotape
- learning by viewing a videotape

Introduction
- introducing the unit

Extra Help
- providing additional study

Print Pathway
- choosing a print alternative

What Lies Ahead
- previewing the unit

Extensions
- going on with the topic

Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts

What You Have Learned
- summarizing what you have learned

Mathematics 31

Course Overview

Mathematics 31 contains 9 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Introduction to Differential Calculus	10%
Unit 2 Differentiation of Algebraic Expressions and Graphing	10%
Unit 3 Practical Application of Derivatives	20%
Unit 4 Integration	10%
Unit 5 Geometric Vectors and Their Application	10%
Unit 6 Algebraic Vectors and Their Application	10%
Unit 7 Inner Product	10%
Unit 8 Systems of Linear Equations	10%
Unit 9 Matrices and Linear Transformations	10%
	100%

Unit Assessment

After completing the unit, you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Integration

This unit covers topics dealing with integration. Each topic contains explanations, examples, and activities to assist you in understanding integration. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in the **Appendix**. In several cases there is more than one way to do the question.

Unit 4 Integration

Contents at a Glance

Value	**Integration**	3
	What You Already Know	5
	Review	5
30%	**Topic 1:** The Integral	6
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 1	
35%	**Topic 2:** The Integral and Areas	27
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 2	
35%	**Topic 3:** Family of Curves	48
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 3	
	Unit Summary	60
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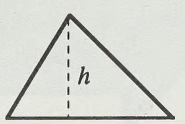
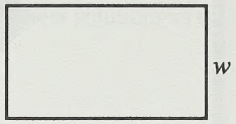
Integration

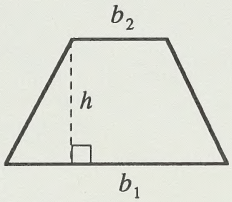
In previous courses you learned methods and formulas for finding areas of plane geometric figures bounded by straight line segments or arcs of circles. Some examples of these figures are shown. In the case of the last figure, previous techniques would be insufficient. In this unit you will consider the question of finding such areas which will lead you to the development of the concept of the **integral**.



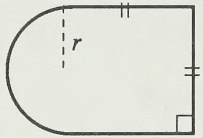
$A = \frac{1}{2}bh$



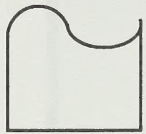
$A = lw$



$A = \frac{1}{2}h(b_1 + b_2)$

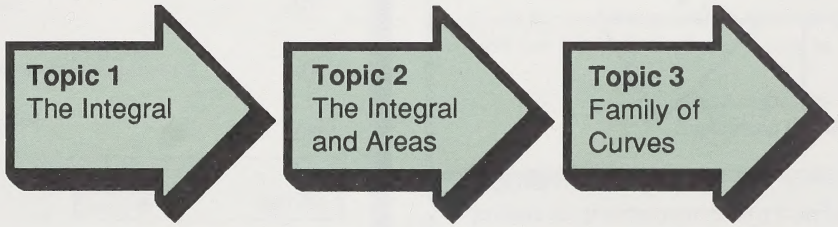


$A = \frac{1}{2}\pi r^2 + 4r^2$



$A = ?$

Unit 4 Integration



What You Already Know

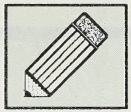
The development of concepts in this unit is dependent on your understanding of concepts reviewed or developed in previous units of this course.

Refresh your memory!

Do you remember the following?

- the definition of derivative
- techniques of differentiation, specifically the power rule and the chain rule
- sketching graphs

Now that you have looked at material you have previously studied, go to the **Review** to confirm your understanding of this material.



Review

1. Find the derivative of the function defined by $y = 2x^5 - 3x^4 + x^3 - 7x^2 + x + 7$.

2. If $y = (5x-4)^4$, find $\frac{dy}{dx}$.

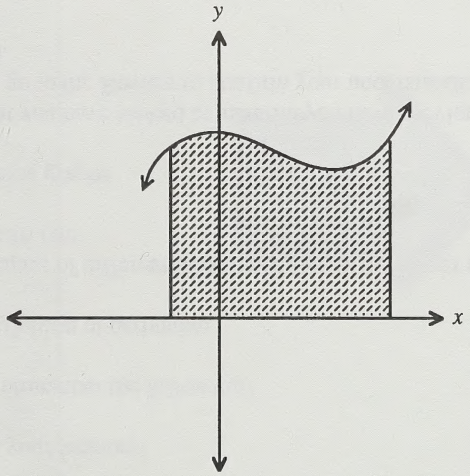
3. If $y = \frac{1}{(4x-3)^3}$, find $\frac{dy}{dx}$.

4. Sketch the graph of $y = 2x^3 - x^2 - 4x + 5$.

Now go to the **Review** solutions in the **Appendix.**

Topic 1 The Integral

Introduction



What is the area of the region shown in the preceding figure? How is the enclosed area related to what you have already done in calculus?

In this unit you will consider these questions and develop the concept and techniques of integration.

What Lies Ahead

Throughout the topic you will learn to

1. approximate the area enclosed between the curve and the x-axis over a defined interval by determining the sum of the areas of a set of rectangles
2. define and determine the integral of a function
3. evaluate definite integrals

Now that you know what to expect, turn the page to begin your study of the integral.

Exploring Topic 1

Activity 1

Approximate the area enclosed between the curve and the x-axis over a defined interval by determining the sum of the areas of a set of rectangles.

In the introduction to this unit, it was mentioned that although there are specific formulas for calculating areas of simple geometric figures, areas bounded by curves other than arcs of circles are difficult to determine. Consideration of this problem will lead to the development of the concepts of **integral calculus**.

Look at the problem of calculating the area under a curve between two limiting values, $x = a$ and $x = b$. The following figure shows such an area that has been divided into six regions of equal width. The area under the curve is the sum of the areas of those six regions.

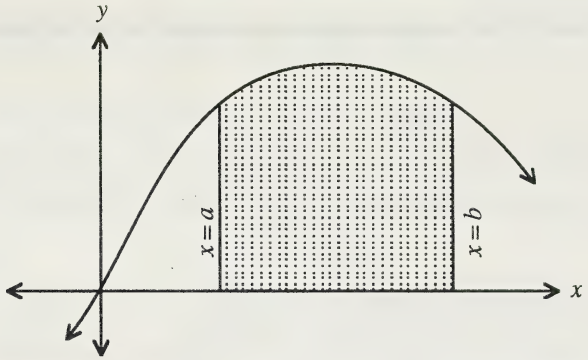


Figure 1

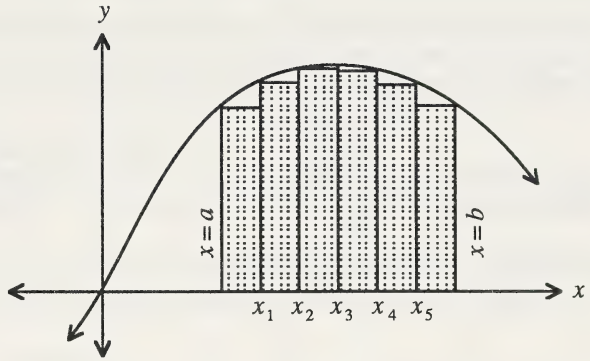


Figure 2

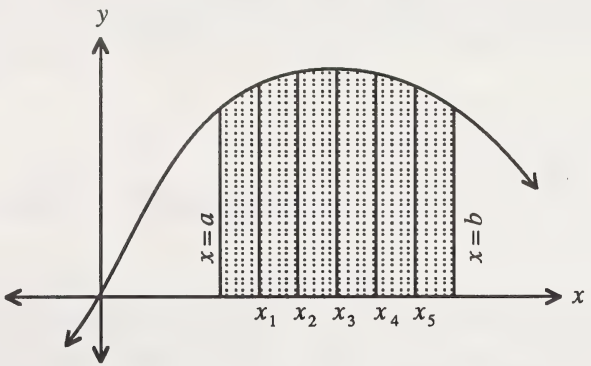
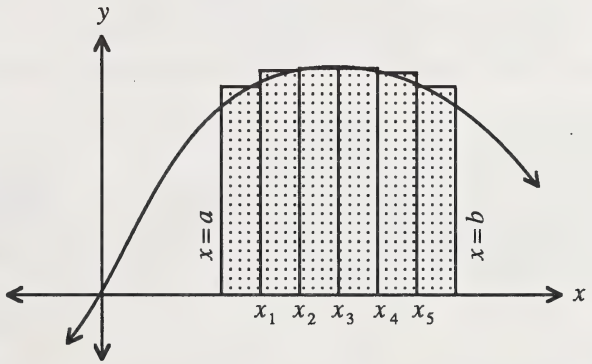


Figure 3



The area of the inscribed rectangles in Figure 1 is less than the area under the curve in Figure 2, which is less than the area of the circumscribed rectangles in Figure 3.

These diagrams show a relationship that will allow you to approximate the area enclosed by the curve, the x-axis, and the lines $x = a$ and $x = b$.

In Figure 1 you have a sequence of six **inscribed** rectangles. The sum of the areas of those rectangles will be less than the actual area.

In Figure 3 you have a sequence of six **circumscribed** rectangles. The sum of the areas of those rectangles will be greater than the actual area in Figure 2.

By calculating the sum of the areas of the **inscribed** rectangles and the sum of the areas of the **circumscribed** rectangles, you arrive at lower and upper bounds between which the actual area must lie.

How can you narrow this interval? Can you find the exact value of the area? If you average the areas of the inscribed and circumscribed rectangles, will that give you the area?

Generally the average of the sums of the areas of the inscribed rectangles and the circumscribed rectangles will not give you the exact area; however, it may give you a good approximation of the area.

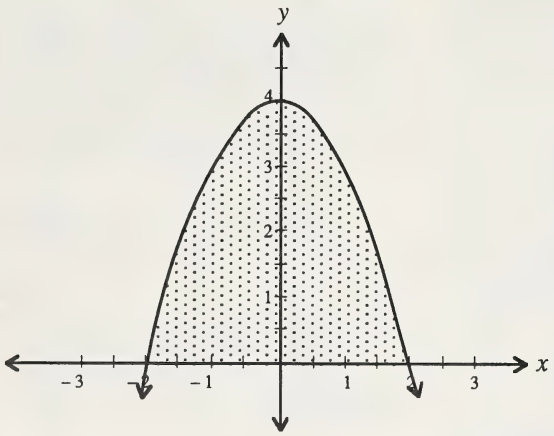
If you decrease the width of your rectangles and increase their number, you will arrive at upper and lower bounds which are closer together, and their average will be a better approximation of the area under the curve. Consider the following example.

Example 1

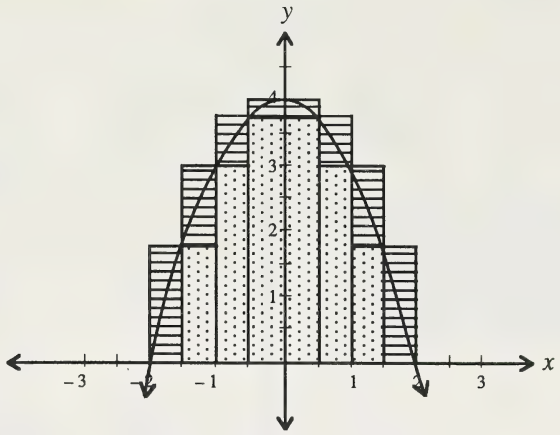
Approximate the area enclosed between the curve defined by $y = 4 - x^2$ and the x-axis.

Solution:

The region defined is shown.



Start by dividing the area into eight regions and constructing the sets of inscribed and circumscribed rectangles.



x	$y = 4 - x^2$
−2.0	0.00
−1.5	1.75
−1.0	3.00
−0.5	3.75
0.0	4.00
0.5	3.75
1.0	3.00
1.5	1.75
2.0	0.00

You need to determine the sum of the areas of each set of rectangles. The area of each rectangle is dependent on a y-coordinate and the difference between two successive x-coordinates. In this case each partition is the same width $(\Delta x = 0.5)$. The y-coordinates can be found using the function defining the curve and then constructing a table of values.

In the interval $-2 \le x \le -1.5$ the inscribed rectangle has a height of 0. However, if you allow this to be a rectangle with an area of zero square units, there will be eight rectangles.

Now consider the eight inscribed rectangles.

The areas will be shown in this chart as they appear on the graph from left to right.

Δx	Height	Area (units^2)
0.5	0.00	0.000
0.5	1.75	0.875
0.5	3.00	1.500
0.5	3.75	1.875
0.5	3.75	1.875
0.5	3.00	1.500
0.5	1.75	0.875
0.5	0.00	0.000
Total Area = 8.500		

Next look at the eight circumscribed rectangles.

Δx	Height	Area (units^2)
0.5	1.75	0.875
0.5	3.00	1.500
0.5	3.75	1.875
0.5	4.00	2.000
0.5	4.00	2.000
0.5	3.75	1.875
0.5	3.00	1.500
0.5	1.75	0.875
Total Area = 12.500		

From these calculations you can safely say that the area you are looking for is between 8.5 units^2 and 12.5 units^2.

Thus, you would expect the area to be approximately 10.5 units^2 (the average of the two areas).

What if you used sixteen partitions instead of eight?

Later in this unit you will learn that the area is exactly $10\frac{2}{3}$ units^2.

The calculations can be summarized in the following tables.

Table 1
$y = 4 - x^2$

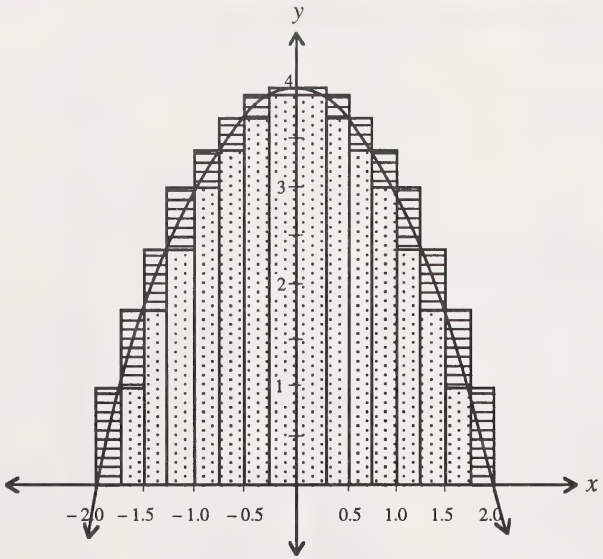
x	$y = 4 - x^2$
−2.00	0.0000
−1.75	0.9375
−1.50	1.7500
−1.25	2.4375
−1.00	3.0000
−0.75	3.4375
−0.50	3.7500
−0.25	3.9375
0.00	4.0000
0.25	3.9375
0.50	3.7500
0.75	3.4375
1.00	3.0000
1.25	2.4375
1.50	1.7500
1.75	0.9375
2.00	0.0000

Table 2
Area of Sixteen
Inscribed Rectangles

Δx	Height	Area (units^2)
0.25	0.0000	0.000 000
0.25	0.9375	0.234 375
0.25	1.7500	0.437 500
0.25	2.4375	0.609 375
0.25	3.0000	0.750 000
0.25	3.4375	0.859 375
0.25	3.7500	0.937 500
0.25	3.9375	0.984 375
0.25	3.9375	0.984 375
0.25	3.7500	0.937 500
0.25	3.4375	0.859 375
0.25	3.0000	0.750 000
0.25	2.4375	0.609 375
0.25	1.7500	0.437 500
0.25	0.9375	0.234 375
0.25	0.0000	0.000 000
	Total Area =	9.625 000

Table 3
Area of Sixteen
Circumscribed Rectangles

Δx	Height	Area (units^2)
0.25	0.9375	0.234 375
0.25	1.7500	0.437 500
0.25	2.4375	0.609 375
0.25	3.0000	0.750 000
0.25	3.4375	0.859 375
0.25	3.7500	0.937 500
0.25	3.9375	0.984 375
0.25	4.0000	1.000 000
0.25	4.0000	1.000 000
0.25	3.9375	0.984 375
0.25	3.7500	0.937 500
0.25	3.4375	0.859 375
0.25	3.0000	0.750 000
0.25	2.4375	0.609 375
0.25	1.7500	0.437 500
0.25	0.9375	0.234 375
	Total Area =	11.625 000

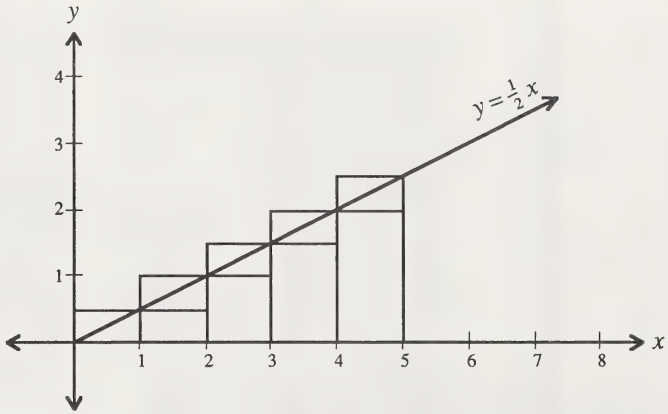


Using these results, the area lies between 9.625 units^2 and 11.625 units^2. The average of these areas is 10.625 units^2 and this is a better approximation of the actual area than the estimate you obtained using eight rectangles.

Now try the following questions.

Use your calculator and do at least one of the following two questions.

1. The graph of $y = \frac{1}{2}x$ for $0 \le x \le 5$ is given as follows.



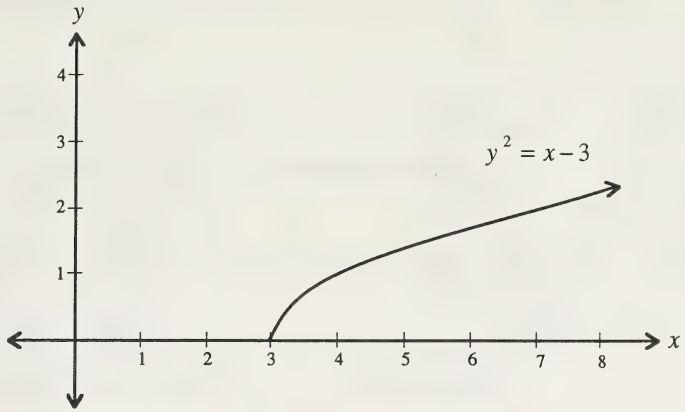
a. Calculate the sum of the areas of five inscribed rectangles and the sum of the areas of five circumscribed rectangles between $y = \frac{1}{2}x$, $x = 0$, $x = 5$, and $y = 0$.

b. If the domain remains unchanged, use ten rectangles to find closer approximations of the areas calculated in question 1a.

c. What is the true area?

2. The graph of $y^2 - x + 3 = 0$ where $y \geq 0$ is given as follows.

$$y^2 = x - 3$$
$$y = +\sqrt{x-3}$$



a. Calculate the sum of the areas of five inscribed rectangles and the sum of the areas of five circumscribed rectangles between $y^2 = x - 3$, $y = 0$, and $x = 8$.

b. If the domain remains unchanged, use ten rectangles to find closer approximations of the areas calculated in question 2a.

c. What is the average area?

For solutions to **Activity 1**, turn to the **Appendix, Topic 1**.

Activity 2

Define and determine the integral of a function.

In the examples and exercises, you have seen that better approximations of the area can be made by taking a greater number of rectangles with a decreased width.

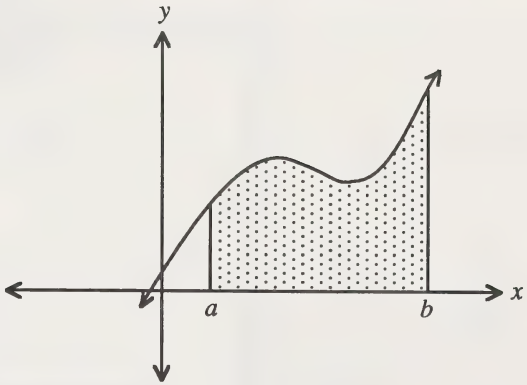
To paraphrase Newton, if you increase the number of rectangles without bound and decrease their widths towards zero, the upper and lower approximations will converge on the actual area.

You have encountered the concept of limit. Can you relate this problem to previous discussions? Is the problem of slope somehow linked to the problem of area?

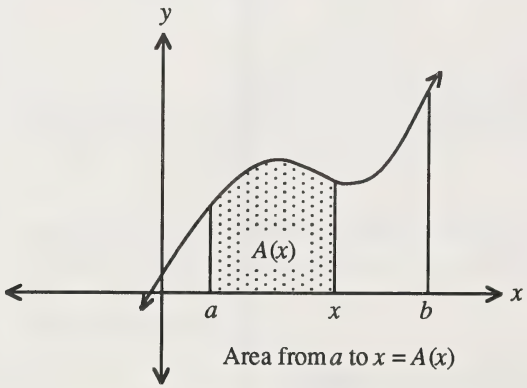
It is the consideration of these questions that links the branches of **differential** and **integral calculus** and leads to the **fundamental theorem of calculus.**

The following discussion presents an argument relating derivatives and area.

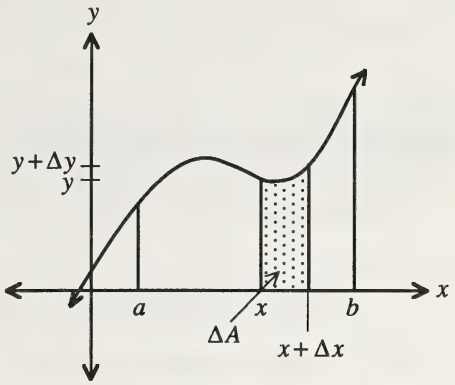
Suppose that $y = f(x)$ is **continuous** and **nonnegative** on the interval $a \le x \le b$. As shown in the following diagram, this defines a region between the curve and the x-axis that has a particular area.



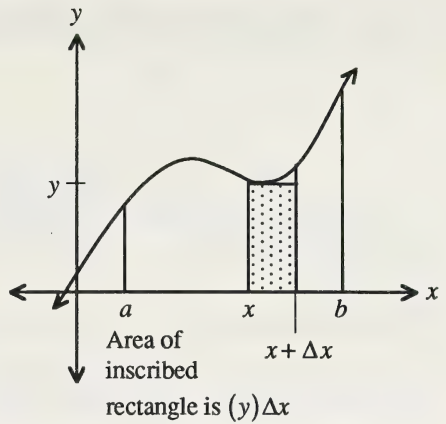
For any value x between a and b, the area under the graph from a to x is dependent on the value of x. This means that the area is some function of x. Call the function $A(x)$.

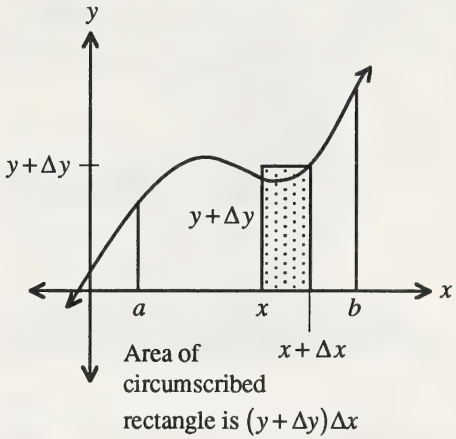


Now if you increase the value of x by Δx, the value of y will increase by some value, Δy, and the area will increase by some value, ΔA.



Using Δx, you can construct an inscribed rectangle and a circumscribed rectangle such that ΔA is greater than the area of the inscribed rectangle and less than that of the circumscribed rectangle.





The area of a rectangle is given by length times width. The width in each case is Δx. The height is a y-value.

You now know that the area of an inscribed rectangle is less than or equal to ΔA. Also, the area of a circumscribed rectangle is greater than or equal to ΔA.

$$y(\Delta x) \le \Delta A \le (y + \Delta y)\Delta x$$

For $\Delta x \neq 0$, you can divide by Δx to get $y \le \frac{\Delta A}{\Delta x} \le y + \Delta y$.

Now let $\Delta x \to 0$.

As $\Delta x \to 0$, $\Delta y \to 0$.

Therefore, $y + \Delta y \to y$.

By definition, $\lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$; therefore,

the inequality statement becomes $y \le \frac{dA}{dx} \le y$

which means $\frac{dA}{dx} = y$.

The result of the preceding discussion has important implications for you. The fact that $y = \frac{dA}{dx}$ shows you that the function that defines the curve is the derivative of the area function.

If you refer back to Example 1, this means $\frac{dA}{dx} = 4 - x^2$.

If this is true, is there a function defining A in terms of x?

To find A, reverse the differentiation procedure. Can you think of how you might do this?

In Example 1, $\frac{dA}{dx} = 4 - x^2$.

If you reverse the power rule, you get $A = 4x - \frac{x^3}{3}$.

Can you think of other expressions for A? They could be $4x - \frac{x^3}{3} + 1$ or $4x - \frac{x^3}{3} - 2$ or $4x - \frac{x^3}{3} + \pi$. How do they differ from $4x - \frac{x^3}{3}$?

Since the derivative of a constant is zero, any other expression for the function defining A can differ from $4x - \frac{x^3}{3}$ only by a constant term.

Note that $4x - \frac{x^3}{3} + c$ is called the **general antiderivative, primitive,** or **integral** of $4 - x^2$, where c is some constant.

If $A = 4x - \frac{x^3}{3}$, then $\frac{dA}{dx} = 4 - x^2$.

So if $\frac{dA}{dx} = 4 - x^2$, then

$A = 4x - \frac{x^3}{3}$.

The relationship between an integral and the area under a curve leads to the symbol for integration. An elongated S, $\int$, is an old form of a printed S. It reminds you of the relationship between integration and the limit of a **sum** of areas.

The integral of $(2x+5)$ with respect to x

$$\int (2x+5)\,dx$$
$$= x^2 + 5x + c$$

To **integrate** or **antidifferentiate** a function, you must reverse the procedure of differentiation. In this unit you will integrate algebraic expressions by reversing the **power rule** and the **chain rule**.

Consider the power rule for differentiation. What steps do you follow?

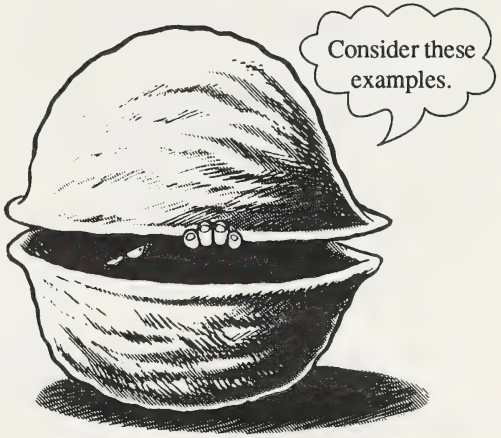
When applying the power rule, you should multiply the coefficient by the exponent and decrease the exponent by one.

To reverse the power rule, add one to the exponent and divide by this new exponent.

Look at the following rule.

If $\frac{dy}{dx} = ax^n$, then

$$y = \int ax^n\,dx$$
$$= \frac{ax^{n+1}}{n+1} + c.$$ (Add one to the exponent, then divide by the new exponent.)



It is important to indicate that it is an integration with respect to x.

To reverse a procedure, you work through the steps in reverse order using inverse operations.

Example 2

Determine $\int x^4\,dx$.

Solution:

$$\int x^4\,dx = \frac{x^{4+1}}{4+1} + c$$
$$= \frac{x^5}{5} + c$$

You might say that ***dx*** is an **integral** part of the symbol for integration. It tells you to determine the integral with respect to x.

Note that neither of the symbols $\int$ or dx is appropriate in your result. It's like the division symbol $(\div)$ in $15 \div 5 = 3$. This symbol indicates the operation of division. The symbol is not part of the result after you have done the division.

When dealing with an expression containing a number of terms, the integral of the expression can be found by determining the integral of each term separately.

Look at the following example.

Example 3

Determine $\int \left(5x^2 + 3\sqrt{x} - \frac{2}{x^2}\right) dx$.

Solution:

$$F(x) = \int \left(5x^2 + 3\sqrt{x} - \frac{2}{x^2}\right) dx$$
$$= \int \left(5x^2 + 3x^{\frac{1}{2}} - 2x^{-2}\right) dx$$
$$= \int 5x^2\,dx + \int 3x^{\frac{1}{2}}\,dx - \int 2x^{-2}\,dx$$
$$= \frac{5x^{2+1}}{2+1} + \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{2x^{-2+1}}{-2+1} + c$$

Therefore, $F(x) = \frac{5}{3}x^3 + 2x^{\frac{3}{2}} + 2x^{-1} + c$.

You will now look at another way of posing the question of integration.

$$\frac{3x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}\left(3x^{\frac{3}{2}}\right)$$
$$= 2x^{\frac{3}{2}}$$

Example 4

Find the general solution of $\frac{dy}{dx} = (3x+5)^2$.

Solution 1:

The expression $\frac{dy}{dx} = (3x+5)^2$ is called a **differential equation**. Its solution is an expression for y and is found by integration.

$$\therefore y = \int (3x+5)^2 \, dx$$

As in differentiation, this integral can be found by expanding and then integrating the expanded form.

$$\begin{aligned} y &= \int (3x+5)^2 \, dx \\ &= \int \left(9x^2 + 30x + 25\right) dx \\ &= \frac{9x^{2+1}}{2+1} + \frac{30x^{1+1}}{1+1} + \frac{25x^{0+1}}{0+1} + c \\ &= 3x^3 + 15x^2 + 25x + c \end{aligned}$$

There are expressions which would be inconvenient or impossible to expand using concepts you know at this point. You are at the same point as you were when you learned to differentiate by the chain rule. Maybe you can reverse this procedure.

Look at Example 4 in another way.

Solution 2:

Given $\frac{dy}{dx} = (3x+5)^2$, let $u = 3x+5$; then, $\frac{du}{dx} = 3$.

The chain rule gives you the following:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\therefore \frac{dy}{du} = \frac{dy}{dx} \div \frac{du}{dx}$$ (The chain rule is reversed.)

Substituting $\frac{dy}{dx} = (3x+5)^2 = u^2$ and $\frac{du}{dx} = 3$, you get the following:

$$\begin{aligned} \frac{dy}{du} &= u^2 \div 3 \\ &= \frac{1}{3}u^2 \end{aligned}$$

$$\begin{aligned} \therefore y &= \int \frac{1}{3}u^2 \, du \\ &= \frac{1}{9}u^3 + c \end{aligned}$$

Since $u = 3x+5$, then $y = \frac{1}{9}(3x+5)^3 + c$.

Study the previous example before you go on. One important restriction must be placed on this procedure. This is shown in the next example.

Example 5

Solve $\frac{dy}{dx} = x\left(x^2 + 5\right)^{\frac{1}{2}}$.

Solution:

Let $u = x^2 + 5$; then, $\frac{du}{dx} = 2x$.

Since $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, then $\frac{dy}{du} = \frac{dy}{dx} \div \frac{du}{dx}$.

Therefore, $\frac{dy}{du} = x\left(u^{\frac{1}{2}}\right) \div (2x)$

$= \frac{x}{2x} u^{\frac{1}{2}}$

$= \frac{1}{2} u^{\frac{1}{2}}$

At this point you must have an expression containing only the variable u.

$$\frac{dy}{du} = \frac{1}{2} u^{\frac{1}{2}} \rightarrow y = \int \frac{1}{2} u^{\frac{1}{2}} du$$

$$= \frac{\frac{1}{2} u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{1}{3} u^{\frac{3}{2}} + c$$

$$= \frac{1}{3}\left(x^2 + 5\right)^{\frac{3}{2}} + c$$

Look at an example where this method will not work.

Example 6

Determine $y = \int \left(m^2 + 3\right)^2 dm$.

Solution:

Let $u = m^2 + 3$; then, $\frac{du}{dm} = 2m$.

Thus, $dm = \frac{du}{2m}$.

Since $\frac{dy}{dm} = \frac{dy}{du} \times \frac{du}{dm}$, then $\frac{dy}{du} = \frac{dy}{dm} \div \frac{du}{dm}$.

$$\int \left(m^2 + 3\right)^2 dm = \int \frac{u^2 du}{2m}$$

$$= \int \frac{1}{2m} u^2 du$$

Note that ***du*** means with respect to u. The expression $\frac{1}{2m} u^2 du$ contains a variable other than u; thus, you cannot integrate with respect to u. Therefore, this method does not work.

In this instance you must expand $\left(m^2 + 3\right)^2$ and then integrate.

$$\int \left(m^2 + 3\right)^2 dm = \int \left(m^4 + 6m^2 + 9\right) dm$$

$$= \frac{1}{5} m^5 + 2m^3 + 9m + c$$

The question of what would happen if you could not expand the expression is left for study in future courses.

It is time for you to practise these methods of integration.

Since integration is with respect to u, the only variable in the expression must be u.

Complete questions 2b, 2c, 2d, 3a, and 3b.
Then do the others if you wish.

1. Expand the second solution for Example 4 and compare it with the first solution. Explain why they are both correct solutions of the differential equation even though they are not identical.

2. Find the general solution for each of the following.

   a. $\frac{dy}{dx} = 3x^4 - 2x^3$

   b. $\frac{dy}{dx} = \frac{1}{4}x^{\frac{2}{5}}$

   c. $\frac{dy}{dx} = 5x^{-4} - \sqrt{3x}$

   d. $\frac{dy}{dx} = \frac{x^3 - 1}{x^5}$

3. Integrate.

   a. $\int (4x-3)^2\,dx$

   b. $\int 2x^3\left(x^4 - 5\right)dx$

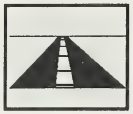
   c. $\int \left[\frac{3x}{\left(x^2-2\right)^4}\right]dx$

   d. $\int \left(\frac{6x^4 - 5x^3 - 20}{x^3}\right)dx$

For solutions to **Activity 2**, turn to the **Appendix, Topic 1**.

$$y = \frac{dA}{dx}$$

Activity 3



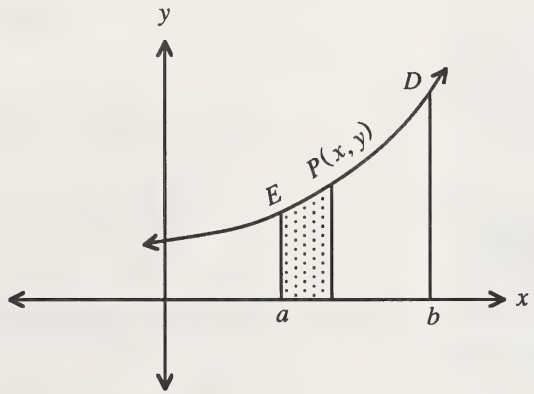
Evaluate definite integrals.

Up to this point you have learned to approximate the area under a curve. The derivative of the area function is the function defining the curve. You also have developed techniques for antidifferentiation (integration).

To bring these elements together, you need the concept of a definite integral.

Fundamental Theorem of Calculus

If a function $y = f(x)$ is continuous on the interval $a \le x \le b$ and $F(x)$ is a function such that $F'(x) = f(x)$ for all x in the interval, then the area $A(x) = F(x) + c$.



Look at the previous diagram.

If the point P coincides with E, then the area $A(x)$ becomes $A(a)$ and is zero.

$$A(a) = 0 = F(a) + c$$
$$\therefore c = -F(a)$$

If the point P coincides with D, then the area $A(x)$ becomes $A(b)$ and is determined as follows:

$$A(b) = F(b) + c$$
$$= F(b) - F(a)$$

The area bounded by $y = f(x)$ and between $x = a$ and $x = b$ is as follows:

$$A(b) - A(a) = F(b) - F(a) - 0$$
$$= F(b) - F(a)$$

Use ${}_aA_b$ to denote this area.

$${}_aA_b = F(b) - F(a)$$

The integral of a function represents the area between the curve and x-axis. (The function that defines the curve is the derivative of the area function.) The symbol $\int_a^b f(x)dx$ is used to denote the area bounded by the curve, the x-axis, $x = a$, and $x = b$.

Therefore, $\int_a^b f(x)dx = F(b) - F(a)$.

The expression $\int_a^b f(x)dx$ is called a **definite integral**.

It is evaluated by determining its primitive $F(x)$ and evaluating the expression $F(b) - F(a)$.

Now look at some examples.

Example 7

Evaluate $\int_0^2 x\,dx$.

Solution:

$$\int_0^2 x\,dx = \left[\frac{x^2}{2}+c\right]_0^2$$ This indicates the upper and lower bounds for evaluation.

$$= \left(\frac{2^2}{2}+c\right)-\left(\frac{0^2}{2}+c\right)$$
$$= (2+c)-(0+c)$$
$$= (2-0)+(c-c)$$
$$= 2$$

In evaluating definite integrals, the constant of integration, c, is the same in both $F(b)$ and $F(a)$. This being the case, the constant is not included when evaluating definite integrals.

Example 8

Evaluate $\int_{-1}^{1}\left[\frac{x}{\left(x^2+4\right)^2}\right]dx$.

Solution:

$$\int\left[\frac{x}{\left(x^2+4\right)^2}\right]dx = \int x\left(x^2+4\right)^{-2}\,dx$$

Let $u = x^2+4$; then, $\frac{du}{dx} = 2x$ and $dx = \frac{du}{2x}$.

$$\int x\left(x^2+4\right)^{-2}\,dx = \int x(u)^{-2}\,\frac{du}{2x}$$
$$= \int \frac{1}{2}u^{-2}\,du$$
$$= -\frac{1}{2}u^{-1}+c$$
$$= -\frac{1}{2}\left(x^2+4\right)^{-1}+c$$

$$\int_{-1}^{1}\left[\frac{x}{\left(x^2+4\right)^2}\right]dx$$
$$= \left[-\frac{1}{2}\left(x^2+4\right)^{-1}\right]_{-1}^{1}$$
$$= -\frac{1}{2}\left(1^2+4\right)^{-1}-\left[-\frac{1}{2}\left((-1)^2+4\right)^{-1}\right]$$
$$= -\frac{1}{10}+\frac{1}{10}$$
$$= 0$$

$$\int x\,dx = \frac{x^2}{2}+c$$

If you have access to a videocassette recorder (VCR), you may wish to view the video titled **Integration**. This program is a good review of the first three activities in this unit. This video is program 10 in the *Catch 31*[1] series.

Now do at least the first four questions in the following exercise.

Evaluate the following.

1. $\int_{-3}^{2}\left(3x^2-4\right)dx$

2. $\int_{7}^{12}(x-3)^{\frac{1}{2}}\,dx$

3. $\int_{-1}^{3}\left[\frac{1}{(3x-2)^2}\right]dx$

4. $\int_{-2}^{1}2x\left(x^2-1\right)^{\frac{1}{2}}\,dx$

5. $\int_{3}^{4}\left(2x^2+3\right)^2\,dx$

6. $\int_{-1}^{2}(5-x)^{\frac{1}{2}}\,dx$

For solutions to **Activity 3**, turn to the **Appendix, Topic 1**.

[1] *Catch 31* is a title of ACCESS Network.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

To integrate a function always add one to the exponent of the variable in each term; then, divide the coefficient of this term by the new exponent. An example is as follows:

$$\int 3x^2\,dx = \frac{3x^{2+1}}{(2+1)}$$
$$= x^3$$

Some questions are not that easy. If you want to find the integral of a power of a polynomial, you need to apply the chain rule. Look at the following example.

Example 9

Integrate $\int 2x\left(x^2-3\right)^5\,dx$.

Solution:

Let $u = x^2 - 3$.

$$du = 2x\,dx$$
$$\therefore dx = \frac{du}{2x}$$

Substitute u and dx into the original question.

$$\int 2x\left(x^2-3\right)^5\,dx = \int (\cancel{2x})u^5\,\frac{du}{(\cancel{2x})}$$
$$= \int u^5\,du$$
$$= \frac{u^6}{6} + c$$
$$= \frac{1}{6}\left(x^2-3\right)^6 + c$$

To evaluate definite integrals in the interval $a \le x \le b$, evaluate the expression $F(b) - F(a)$.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example 10

Evaluate $\int_3^5 (3x+1)\,dx$.

Solution:

$$\int_3^5 (3x+1)\,dx = \left[\frac{3x^2}{2}+x\right]_3^5$$

$$= \left[\frac{3(5)^2}{2}+(5)\right] - \left[\frac{3(3)^2}{2}+(3)\right]$$

$$= \frac{75}{2}+5-\frac{27}{2}-3$$

$$= 26$$

Do the following questions.

1. Integrate the following.

   a. $\int (3x^2-4x+1)\,dx$

   b. $\int (x+1)(x-3)\,dx$

   c. $\int \left[\frac{1}{(x-3)^2}\right] dx$

   d. $\int \left(\sqrt{x+2}\right) dx$

2. Evaluate the following.

   a. $\int_1^5 3x\,dx$

   b. $\int_{-1}^3 \left(4x^{-2}+1\right) dx$

For solutions to **Extra Help**, turn to the **Appendix, Topic 1**.

Extensions

If you are looking for more challenging explorations, try the following questions.

1. $\int \left[\left(x^2-\frac{1}{x^2}\right)^2+4\right]^{\frac{1}{2}} dx$

2. $\int \left[\frac{x^3+x^2-5x}{(x-1)^2}\right] dx$

For solutions to **Extensions**, turn to the **Appendix, Topic 1**.

Topic 2 The Integral and Areas

Introduction

In Example 1 of the previous topic you started by approximating the area enclosed between the x-axis and the curve defined by $y = 4 - x^2$. At that time it was indicated that the exact value was $10\frac{2}{3}$ units^2. In this topic you will learn how to find the exact area.

What Lies Ahead

Throughout the topic you will learn to

1. determine the area between a curve and the x-axis over a given domain
2. determine the area enclosed between the curves defined by two functions

Now that you know what to expect, turn the page to begin your study of the integral and areas.

Exploring Topic 2

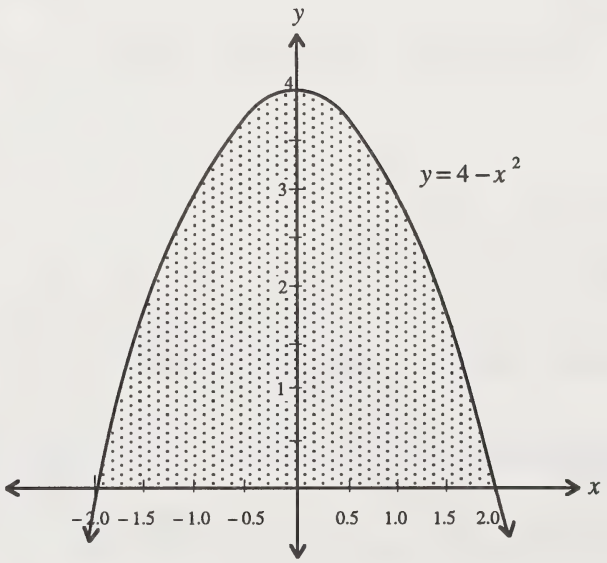
Activity 1

Determine the area between a curve and the x-axis over a given domain.

From the discussion in the previous topic you know that if $y = f(x)$ is **continuous** and **positive** on an interval, then $y = \frac{dA}{dx}$, where A is the area between the graph and the x-axis over an interval.

This gives you $A = \int y\,dx$, where $y = f(x)$ is the defining equation of the curve.

In the first example of the previous topic, $y = 4 - x^2$.



This gives you $A = \int (4 - x^2)\,dx$; thus, $A = 4x - \frac{x^3}{3} + c$ is the area function.

The limits on the area are the axis-intercepts, $x = \pm 2$.

To determine this area, evaluate the following.

$$
\begin{aligned}
{}_{-2}A_2 &= \int_{-2}^{2} \left(4 - x^2\right) dx \\
&= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\
&= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) \\
&= \frac{32}{3} \\
&= 10\frac{2}{3}
\end{aligned}
$$

The area is $10\frac{2}{3}$ units^2.

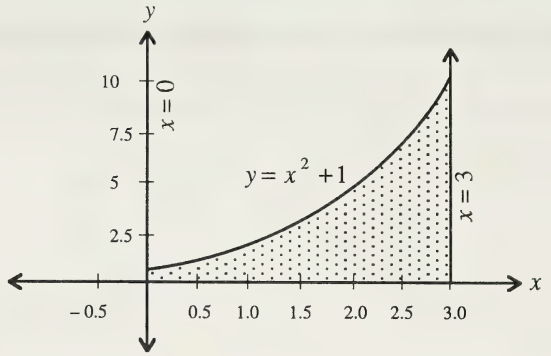
It is important to note that this is based on the function being continuous and positive on this interval.

Example 1

Determine the area bounded by $y = x^2 + 1$, the x-axis, and the lines $x = 0$ and $x = 3$.

Solution:

The curve $y = x^2 + 1$ is continuous and positive on the interval $0 \le x < 3$ as shown in the diagram.



$$
\begin{aligned}
{}_0A_3 &= \int_0^3 \left(x^2 + 1\right) dx = \left[\frac{x^3}{3} + x\right]_0^3 \\
&= (9 + 3) - (0 + 0) \\
&= 12
\end{aligned}
$$

The area is 12 units^2.

Is the area always equal to the definite integral?

If you think the area is equal to the definite integral only when the function is continuous and positive, you are correct.

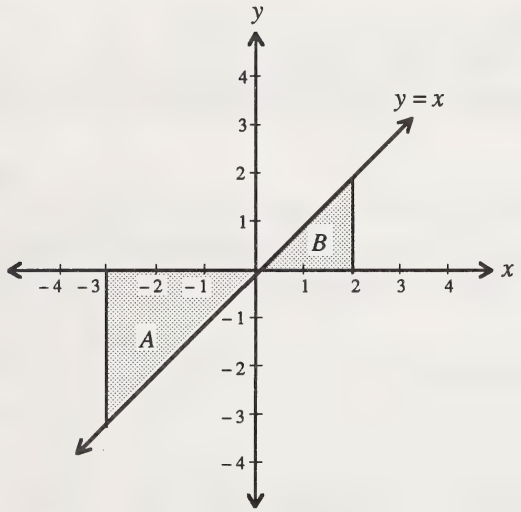
What about regions below the x-axis on the interval or if the graph crosses the axis on the interval?

Investigate an example to see if you can determine the area in these cases.

Example 2

Determine the area enclosed betweeen the x-axis and the line defined by $y = x$ on the interval $-3 \le x \le 2$.

Solution:



Using the formula for the area of a triangle, the area of region A is 4.5 units^2 and the area of region B is 2 units^2. The total area is 6.5 units^2.

If you evaluate $\int_{-3}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{-3}^{2}$, you get a value of -2.5.

What happened? How can you fix it?

Look at the interval where the region is below the x-axis.

$$\int_{-3}^{0} x\,dx = \left[\frac{x^2}{2}\right]_{-3}^{0}$$

$$= \frac{0^2}{2} - \frac{(-3)^2}{2}$$

$$= -4.5$$

That is where you will find the problem.

Regions below the x-axis correspond to negative evaluations for the definite integral.

You can fix that by using absolute value.

$$_{-3}A_0 = \left|\int_{-3}^{0} x\,dx\right|$$

$$= |-4.5|$$

$$= 4.5$$

$$_0A_2 = \int_0^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= 2$$

Therefore, the total area is $_{-3}A_0 + {}_0A_2 = 4.5 + 2 = 6.5$ units^2.

It is critical to know where the curve is above the x-axis and where it is below the x-axis on a given interval. A sketch of the function is helpful when determining this.

Go to the next example.

Example 3

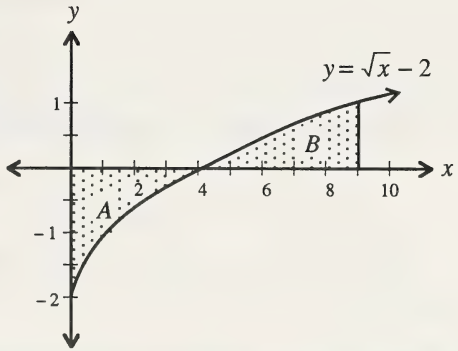
Find the area bounded by $y=\sqrt{x}-2$ and the x-axis over the interval $0\le x\le 9$.

Solution:

First sketch the curve noting particularly any x-intercept in the interval $0\le x\le 9$. Shade the region where the area is to be determined.

To find the x-intercept, let $y=0$.

$$y=\sqrt{x}-2$$
$$0=\sqrt{x}-2$$
$$\sqrt{x}=2$$
$$x=4$$



Region A is below the x-axis.

$$
\begin{aligned}
{}_0A_4 &= \left|\int_0^4 \left(x^{\frac{1}{2}}-2\right)dx\right| \\
&= \left|\left[\frac{2}{3}x^{\frac{3}{2}}-2x\right]_0^4\right| \\
&= \left|\left[\frac{2}{3}(4)^{\frac{3}{2}}-2(4)\right]-\left[\frac{2}{3}(0)^{\frac{3}{2}}-2(0)\right]\right| \\
&= \left|\frac{16}{3}-8\right| \\
&= \left|\frac{-8}{3}\right| \\
&= \frac{8}{3}
\end{aligned}
$$

Region B is above the x-axis.

$$
\begin{aligned}
{}_4A_9 &= \int_4^9 \left(x^{\frac{1}{2}}-2\right)dx \\
&= \left[\frac{2}{3}x^{\frac{3}{2}}-2x\right]_4^9 \\
&= \left[(18-18)-\left(\frac{16}{3}-8\right)\right] \\
&= \left(0-\frac{-8}{3}\right) \\
&= \frac{8}{3}
\end{aligned}
$$

The total area is ${}_0A_4+{}_4A_9=\frac{8}{3}+\frac{8}{3}=\frac{16}{3}$ units ^{2}.

No matter how complex the function, you can follow the procedure shown in Example 3. Consider the next example.

Example 4

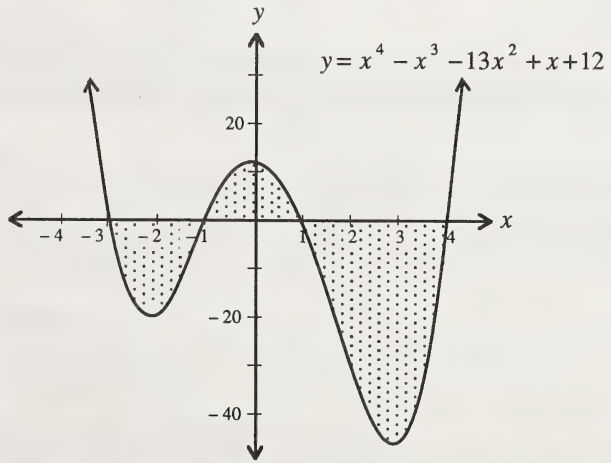
Find the area of the regions bounded by the x-axis and the curve defined by $y = x^4 - x^3 - 13x^2 + x + 12$.

Solution:

Using the factor theorem and synthetic division, determine the roots of the function.

$$y = x^4 - x^3 - 13x^2 + x + 12$$
$$= (x-1)(x+1)(x+3)(x-4)$$

The graph crosses the x-axis at $x = -3$, $x = -1$, $x = 1$, and $x = 4$. Using your knowledge about polynomial functions, sketch the graph.



In the graph two regions are below the x-axis and one region is above the x-axis.

$$-3A_{-1} = \left| \int_{-3}^{-1} \left(x^4 - x^3 - 13x^2 + x + 12\right) dx \right|$$
$$= \left| \left[\frac{x^5}{5} - \frac{x^4}{4} - \frac{13x^3}{3} + \frac{x^2}{2} + 12x \right]_{-3}^{-1} \right|$$
$$= \left| -24.2\overline{6} \right|$$
$$= 24.2\overline{6}$$

$$-1A_{1} = \int_{-1}^{1} \left(x^4 - x^3 - 13x^2 + x + 12\right) dx$$
$$= \left[\frac{x^5}{5} - \frac{x^4}{4} - \frac{13x^3}{3} + \frac{x^2}{2} + 12x \right]_{-1}^{1}$$
$$= 15.7\overline{3}$$

$$1A_{4} = \left| \int_{1}^{4} \left(x^4 - x^3 - 13x^2 + x + 12\right) dx \right|$$
$$= \left| \left[\frac{x^5}{5} - \frac{x^4}{4} - \frac{13x^3}{3} + \frac{x^2}{2} + 12x \right]_{1}^{4} \right|$$
$$= 88.65$$

Total area $= {}_{-3}A_{-1} + {}_{-1}A_{1} + {}_{1}A_{4}$
$$= 24.2\overline{6} + 15.7\overline{3} + 88.65$$
$$\doteq 128.65$$

Therefore, the total area is about 128.65 units^2.

Now try the following questions. Do at least the first two.

1. Determine the area enclosed by the x-axis, the graph of $y = f(x)$, and the two vertical lines given.

   a. $f(x) = x - 3,\ x = 2,\ x = 4$

   b. $f(x) = x^3 - 4x^2,\ x = -3,\ x = 3$

   c. $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}},\ x = 0,\ x = 3$

2. Determine the area enclosed by the x-axis and the graph of y defined as follows.

   a. $y = -2x^2 + x + 3$

   b. $y = 3x^2 - x^3$

3. Determine the area enclosed by the x-axis, the graph of $y = f(x)$, and the two vertical lines given.

   a. $y = -2x^2 - 3x + 2,\ x = -2,\ x = 1$

   b. $y = \sqrt{x - 1},\ x = 2,\ x = 5$

4. Determine the area enclosed by the x-axis and the graph of y defined as follows.

   a. $y = x^4 - 9x^2$

   b. $y = (x - 2)^2 - 3$

For the solutions to **Activity 1**, turn to the **Appendix, Topic 2**.

Activity 2

Determine the area enclosed between the curves defined by two functions.

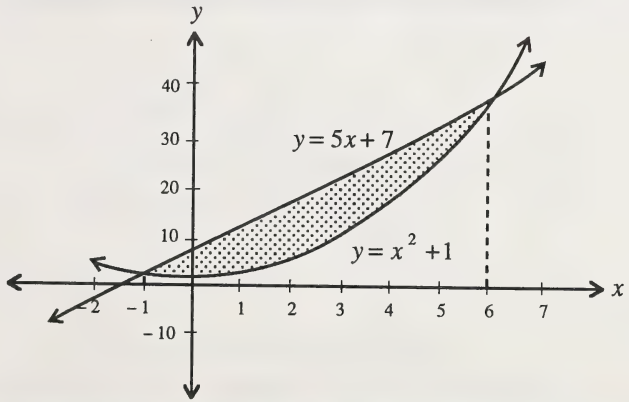
After completing the questions in Activity 1, you should feel comfortable with the procedure for calculating the area enclosed between a curve and the x-axis. Now you will investigate the problem of determining the area between two curves.

Example 5

Determine the area between the curves defined by $y = x^2 + 1$ and $y = 5x + 7$.

Solution:

Sketch the curves and determine the region involved.



From the diagram the points of intersection of the graphs of the functions are $(-1, 2)$ and $(6, 37)$.

$$y = 5x + 7$$
$$y = x^2 + 1$$

At the points of intersection the y-values will be the same.

$$y = y$$
$$5x + 7 = x^2 + 1$$
$$x^2 - 5x - 6 = 0$$
$$(x - 6)(x + 1) = 0$$
$$x - 6 = 0 \text{ or } x + 1 = 0$$
$$x = 6 \qquad x = -1$$

The corresponding y-values are determined by substitution into either of the defining equations.

When $x = 6$, $y = 37$.
When $x = -1$, $y = 2$.

Now, $\int_{-1}^{6} (5x + 7)\,dx$ is the area bounded by the line and the x-axis over the interval $-1 \le x \le 6$, and $\int_{-1}^{6} \left(x^2 + 1\right)dx$ is the area under the parabola over that same interval.

How can you determine the area between the two curves?

If you said subtract, you are right. The area between these curves is given by the following:

$$\int_{-1}^{6}(5x+7)dx-\int_{-1}^{6}\left(x^2+1\right)dx$$

$$A=\int_{-1}^{6}\left[(5x+7)-\left(x^2+1\right)\right]dx$$
$$=\int_{-1}^{6}\left(-x^2+5x+6\right)dx$$

This simplification was done using the first rule in the margin.

$$A=\left[\frac{-x^3}{3}+\frac{5x^2}{2}+6x\right]_{-1}^{6}$$
$$=57\frac{1}{6}$$

The area is $57\frac{1}{6}$ units^2.

When you encounter something new, you should try to think about what other possibilities there are. However, in Example 5 the position of the region relative to the x-axis does not affect the area or the way you calculate it. This is not always the case.

Consider the following example where the region goes below the x-axis.

Two integral rules are as follows:

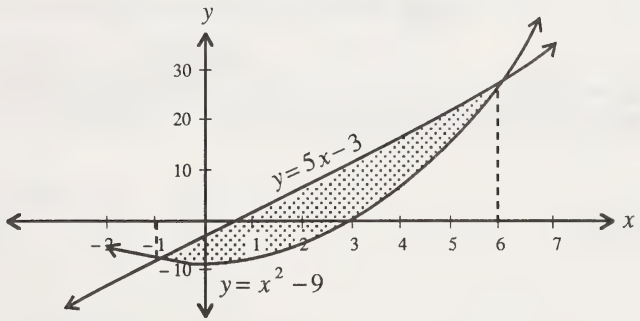
- $\int[f(x)\pm g(x)]dx$
  $=\int f(x)dx\pm\int g(x)dx$
- $\int[kf(x)]dx=k\int f(x)dx$

Example 6

Determine the area between the curves defined by $y = x^2 - 9$ and $y = 5x - 3$.

Solution:

Sketch the graph.



How does this compare with the graph in Example 5? It is the same, except it has been shifted down 10 units.

You are dealing with a region which has been **translated** 10 units down on the coordinate plane. The area of the region is not changed by this translation, so it should be $57\frac{1}{6}$ units^2 as it was in Example 5.

The points of intersection occur where $y = y$.

$$x^2 - 9 = 5x - 3$$
$$x^2 - 5x - 6 = 0$$
$$(x-6)(x+1) = 0$$
$$x = 6 \text{ or } x = -1$$

These are the same x-coordinates as in Example 5.

By substitution you get $(6, 27)$ and $(-1, -8)$ as the points of intersection.

$$\int_{-1}^{6}(5x-3)\,dx-\int_{-1}^{6}\left(x^2-9\right)dx=\int_{-1}^{6}\left[(5x-3)-\left(x^2-9\right)\right]dx$$

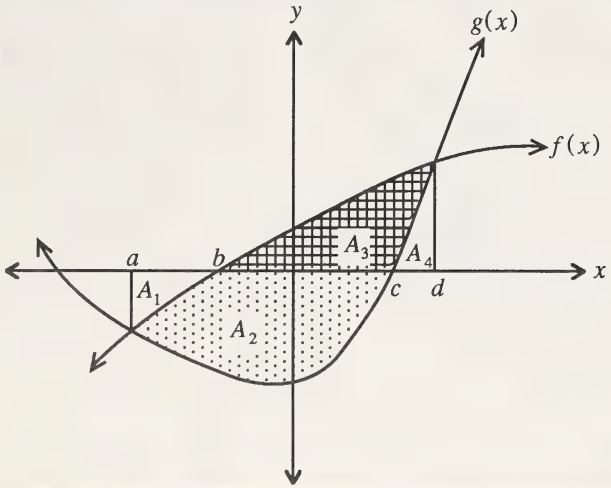
$$=\int_{-1}^{6}\left(-x^2+5x+6\right)dx$$

$$=57\frac{1}{6}$$

The enclosed area is $57\frac{1}{6}$ units^2.

You can ignore the position of the curves relative to the x-axis when you are looking for the area between two curves.

Look at a more general case to justify this observation.

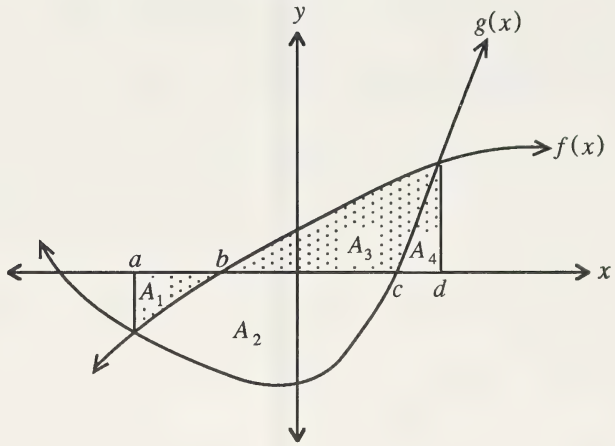


In the diagram there are two curves defined by $y=f(x)$ and $y=g(x)$ which intersect where $x=a$ and $x=d$. The area of the region between the curves is given by A_2+A_3. In the interval $a<x<d$, $f(b)=0$ and $g(c)=0$; that is, both curves cross the x-axis.

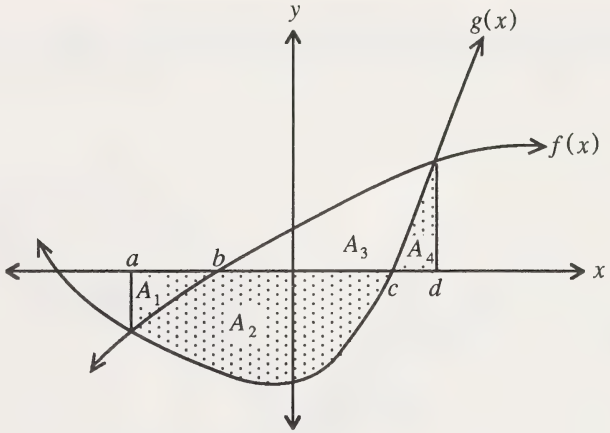
Recall the relationship between the definite integral and area.

- If $f(x)>0$ on the interval $a<x<b$, then $\int_a^b f(x)\,dx={}_aA_b$.
- If $f(x)<0$ on the interval $a<x<b$, then $\int_a^b f(x)\,dx=-{}_aA_b$.

Use this relationship and the diagram.



$$\int_a^d f(x)\,dx=-A_1+A_3+A_4$$



$$\int_a^d g(x)dx = -A_1 - A_2 + A_4$$

You want to find the area between the curves which is $A_3 + A_2$.

$$\int_a^d f(x)dx - \int_a^d g(x)dx = (-A_1 + A_3 + A_4) - (-A_1 - A_2 + A_4)$$
$$= A_3 + A_2$$

In general if the curves defined by $y = f(x)$ and $y = g(x)$ are such that in the interval $a < x < b$, $f(x) > g(x)$, then the area between the curves over that interval is as follows:

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

You take the integral of the top curve minus the bottom curve over the interval.

The expression $f(x) > g(x)$ means that $y = f(x)$ defines the top curve and $y = g(x)$ defines the bottom curve.

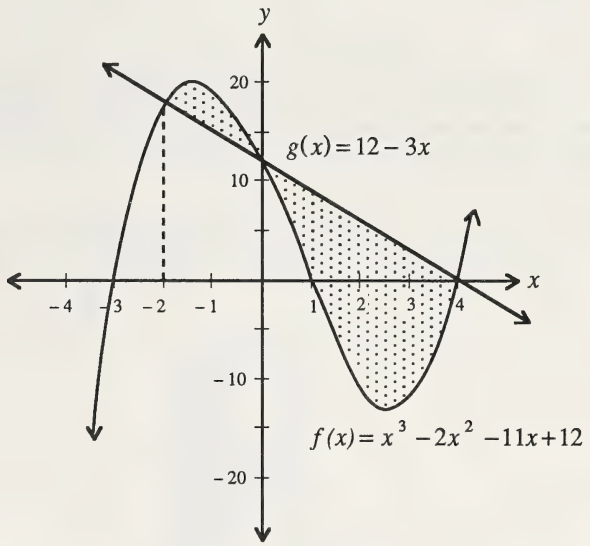
Look at one more example.

Example 7

Find the total area between the curves defined by $f(x)=x^3-2x^2-11x+12$ and $g(x)=12-3x$.

Solution:

As shown in the graph there are two regions.



The curves intersect when $f(x)=g(x)$.

$$x^3-2x^2-11x+12=12-3x$$
$$x^3-2x^2-8x=0$$
$$x(x+2)(x-4)=0$$
$$x=0 \text{ or } x=-2 \text{ or } x=4$$

The points of intersection are $(-2,18)$, $(0,12)$, and $(4,0)$.

For $-2<x<0$, $f(x)>g(x)$ where $f(x)$ defines the top curve.

$$-_2A_0=\int_{-2}^{0}[f(x)-g(x)]dx$$
$$-_2A_0=\int_{-2}^{0}\left[\left(x^3-2x^2-11x+12\right)-(12-3x)\right]dx$$
$$=\int_{-2}^{0}\left(x^3-2x^2-8x\right)dx$$
$$=\left[\frac{x^4}{4}-\frac{2x^3}{3}-4x^2\right]_{-2}^{0}$$
$$=\frac{20}{3}$$

For $0 < x < 4$, $g(x)$ defines the top curve.

$$
\begin{aligned}
{}_0A_4 &= \int_0^4 [g(x) - f(x)]\,dx \\
&= \int_0^4 \left[(12 - 3x) - \left(x^3 - 2x^2 - 11x + 12\right)\right] dx \\
&= \int_0^4 \left(-x^3 + 2x^2 + 8x\right) dx \\
&= \left[\frac{-x^4}{4} + \frac{2x^3}{3} + 4x^2\right]_0^4 \\
&= \frac{128}{3}
\end{aligned}
$$

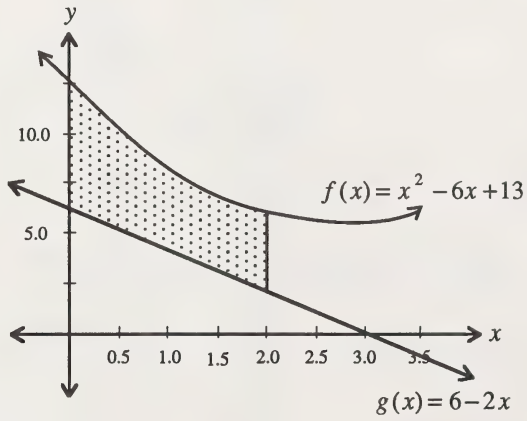
The total area is $\frac{20}{3} + \frac{128}{3} = \frac{148}{3} = 49\frac{1}{3}$ units^2.

It is important to know if the curves intersect in an interval, and if so it is important to know which is the **top** curve and the **bottom** curve on that interval. The curves need not intersect if an interval is specified as illustrated in the following example.

Example 8

Find the area bounded by the curve defined by $f(x) = x^2 - 6x + 13$ and the line defined by $g(x) = 6 - 2x$ over the interval $0 \le x \le 2$.

Solution:

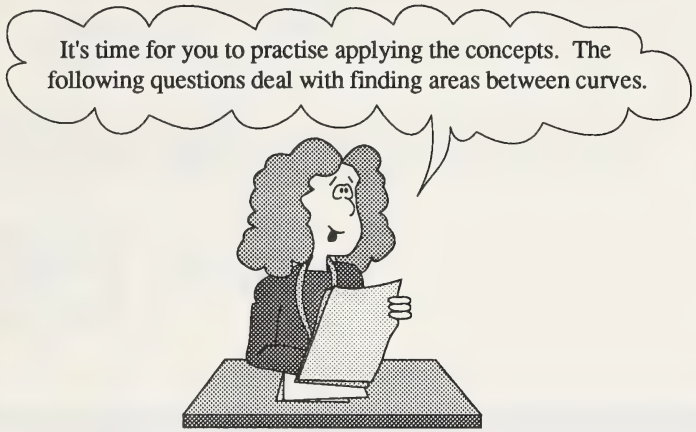


As shown by the graph, the curves do not intersect, but the region is well-defined using the given curves and the vertical lines $x = 0$ and $x = 2$.

For $0 \le x \le 2$, $f(x) > g(x)$. The area is given by the following:

$$
\begin{aligned}
{}_0A_2 &= \int_0^2 [f(x) - g(x)]\,dx \\
&= \int_0^2 \left[\left(x^2 - 6x + 13\right) - (6 - 2x)\right] dx \\
&= \int_0^2 \left(x^2 - 4x + 7\right) dx \\
&= \left[\frac{x^3}{3} - 2x^2 + 7x\right]_0^2 \\
&= \frac{26}{3}
\end{aligned}
$$

The area is $\frac{26}{3}$ units^2.



Do at least the first three questions.

1. Find the area between the curves defined by $y = -x^2 + 4$ and $y = 2x + 1$.

2. Find the area between the curves defined by $y = x^3 - 5x^2 + 2x + 8$ and $y = -2x + 8$.

3. Find the area bounded by $y = x^2 + 4x - 9$ and the line defined by $y = 2x - 1$ over the interval $-3 \le x \le 1$.

4. Find the area bounded by $y = 3x + 1$ and $y = \sqrt{x}$ over the interval $0 \le x \le 5$.

5. Find the area enclosed by the graphs of $y = x^2 - 2x + 5$ and $y = -x^2 + 4x + 1$.

For solutions to **Activity 2**, turn to the **Appendix**, **Topic 2**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

The area of the region bounded by the function $f(x) = y$ and the x-axis over the interval $a \le x \le b$ is the integral of y.

$$A = \left| \int_a^b f(x)\,dx \right|$$

If the region is above the x-axis, the area is positive; if the region is under the x-axis, the area is negative. The absolute value sign would make the area of any region positive. The fundamental theorem of integral calculus states the following:

$\int_a^b f(x)\,dx = F(b) - F(a)$, where F is a function such that the following can be stated:

$$\frac{dF}{dx} = f(x)$$

You can evaluate definite integrals by applying this theorem.

Example 9

Evaluate $\int_1^4 (x+3)\,dx$.

Solution:

$$\begin{aligned}\int_1^4 (x+3)\,dx &= \left[\frac{x^2}{2} + 3x\right]_1^4 \\ &= \left[\frac{4^2}{2} + 3(4)\right] - \left[\frac{1^2}{2} + 3(1)\right] \\ &= 8 + 12 - \frac{1}{2} - 3 \\ &= 16\frac{1}{2}\end{aligned}$$

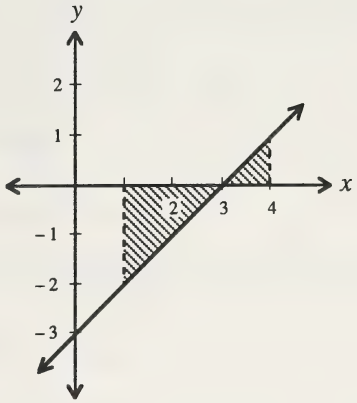
If the curve intersects the x-axis, you have to calculate the areas separately. The total area is the sum of the areas of the regions separated by the point(s) of intersection over the required interval. The total area is the sum of the absolute values of these regions.

Example 10

Evaluate $\int_1^4 (x-3)\,dx$.

Solution:

If $y = x - 3$, then the x-intercept is 3.



$$\text{Total area} = {}_1A_3 + {}_3A_4$$

$$= \left|\int_1^3 (x-3)\,dx\right| + \int_3^4 (x-3)\,dx$$

$$= \left|\left[\frac{x^2}{2} - 3x\right]_1^3\right| + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$= \left|\left[\frac{3^2}{2} - 3(3)\right] - \left[\frac{1^2}{2} - 3(1)\right]\right| + \left[\frac{4^2}{2} - 3(4)\right] - \left[\frac{3^2}{2} - 3(3)\right]$$

$$= \left|\frac{9}{2} - 9 - \frac{1}{2} + 3\right| + \left(\frac{16}{2} - 12\right) - \left(\frac{9}{2} - 9\right)$$

$$= |-2| + \frac{1}{2}$$

$$= 2\frac{1}{2}$$

To determine the area between two curves, you have to know the points of intersection. A graph would be very useful. A graph helps you to identify the intervals you should be using. If the two curves do not intersect, then you have to know the interval which indicates the vertical boundary of the region. The area of the region is the integral of the difference of the two functions. Use the function of the top curve; then, subtract the function of the bottom curve from this.

Example 11

Find the total area between the curves defined by $y = x^3 - 11x + 2$ and $y = -2x + 2$.

Solution:

$y = x^3 - 11x + 2$ (1)

$y = -2x + 2$ (2)

(1) = (2): $x^3 - 11x + 2 = -2x + 2$

$$x^3 - 9x = 0$$

$$x(x^2 - 9) = 0$$

$x = 0$ or $x = 3$ or $x = -3$

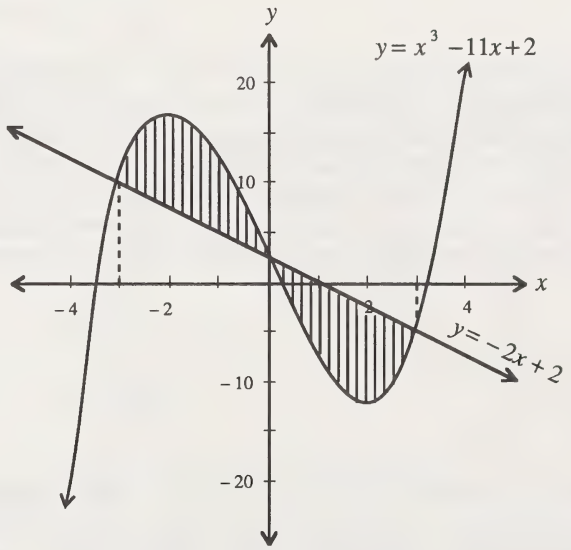
Substitute in (2).

At $x = 0$, $y = 2$.
At $x = 3$, $y = -4$.
At $x = -3$, $y = 8$.

The points of intersection are $(0, 2)$, $(3, -4)$, and $(-3, 8)$.

The following chart lists values for x and y in the equation $y = x^3 - 11x + 2$.

x	-4	-3	-2	0	1	4
y	-18	8	16	2	-8	22



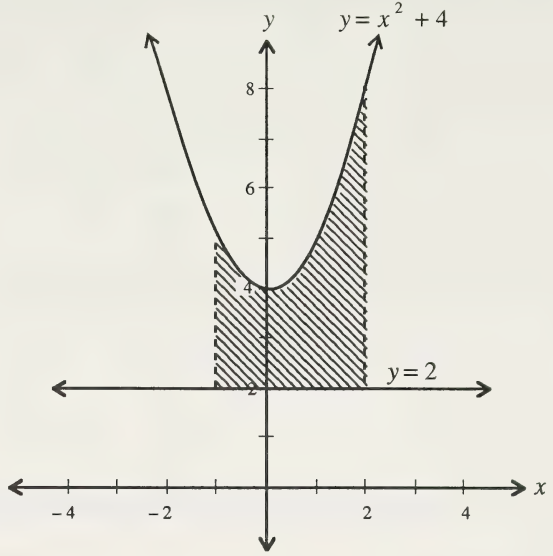
$$\text{Total area} = \int_{-3}^{0}\left[\left(x^3 - 11x + 2\right) - \left(-2x + 2\right)\right]dx$$
$$+\int_{0}^{3}\left[\left(-2x + 2\right) - \left(x^3 - 11x + 2\right)\right]dx$$
$$= \int_{-3}^{0}\left(x^3 - 9x\right)dx + \int_{0}^{3}\left(-x^3 + 9x\right)dx$$
$$= \left[\frac{x^4}{4} - \frac{9x^2}{2}\right]_{-3}^{0} + \left[\frac{-x^4}{4} + \frac{9x^2}{2}\right]_{0}^{3}$$
$$= 0 - \left(\frac{81}{4}\right) + \frac{81}{2} + \frac{-(3)^4}{4} + \frac{9(3)^2}{2} - 0$$
$$= -\frac{81}{4} + \frac{162}{4} - \frac{81}{4} + \frac{162}{4}$$
$$= \frac{162}{4}$$
$$= 40\frac{1}{2}$$

If the two curves do not intersect, then you have to know the interval.

Example 12

Find the area bounded by the curve defined by $y = x^2 + 4$ and the line defined by $y = 2$ over the interval $-1 \le x \le 2$.

Solution:



$$\int_{-1}^{2}\left[\left(x^2 + 4\right) - 2\right]dx = \int_{-1}^{2}\left(x^2 + 2\right)dx$$
$$= \left[\frac{x^3}{3} + 2x\right]_{-1}^{2}$$
$$= \frac{2^3}{3} + 2(2) - \frac{(-1)^3}{3} - 2(-1)$$
$$= \frac{8}{3} + 4 + \frac{1}{3} + 2$$
$$= 9$$

If the two curves intersect and the points of intersection do not represent the left- or right-hand extremes of the region, then you have to be very careful when you separate the graph into smaller regions. If you wish to see an example of this kind, please turn to the **Extensions** section in this topic.

Do the following questions.

1. Find the area between the graph of $y = x^2 + 3$ and the x-axis from $x = -2$ to $x = 2$.

2. Find the area between the curves $y = x^2 - 2$ and $y = -x^2 + 2$.

3. Find the area between the curves $y = -x^2 - 1$ and $y = 1$ over the interval $-1 \le x \le 1$.

For solutions to **Extra Help**, turn to the **Appendix, Topic 2**.

Extensions

If two curves are given, the points of intersection of these two curves may not represent the left- or right-hand extremes of the region. If you want to find the area enclosed by these two curves, then you have a new type of problem.

Example 13

Find the area between the graphs of $y^2 = x + 3$ and $y = x + 1$.

Solution:

The points of intersection must be determined.

$$y^2 = x + 3 \quad (1)$$
$$y = x + 1 \quad (2)$$

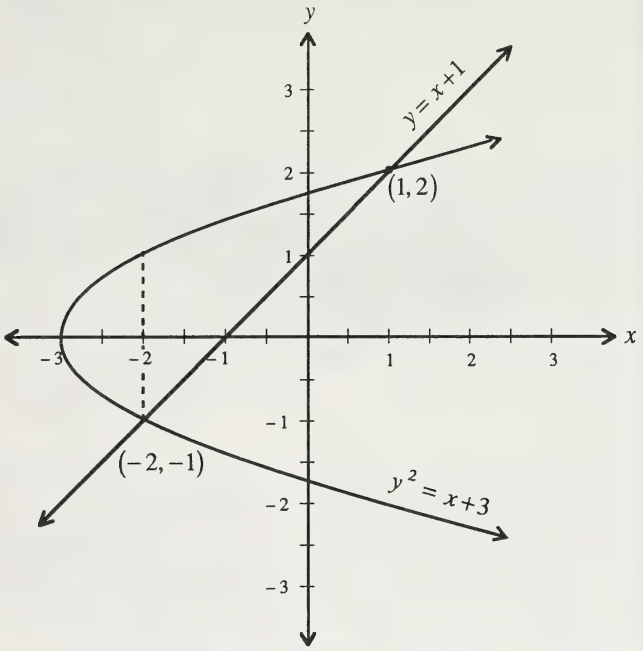
$$y^2 = y^2$$
$$\therefore (x+1)^2 = x + 3$$
$$x^2 + 2x + 1 = x + 3$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$

$$x = -2 \text{ or } x = 1$$

At $x = -2$, $y = -1$.

At $x = 1$, $y = 2$.

Thus, $(-2,-1)$ and $(1,2)$ are the two points of intersection.



The extremes are $(-3,0)$ and $(1,2)$. You have to separate the region over $-3 \le x \le -2$ into two parts. The top half of $y^2 = x+3$ is $y = \sqrt{x+3}$ and the bottom half of $y^2 = x+3$ is $y = -\sqrt{x+3}$; thus, the area can be determined between $y = \sqrt{x+3}$ and $y = -\sqrt{x+3}$ over the interval $-3 \le x \le -2$. Then the area can be calculated between $y = \sqrt{x+3}$ and $y = x+1$ for the interval $-2 \le x \le 1$.

$$A = \int_{-3}^{-2}\left[\sqrt{x+3} - \left(-\sqrt{x+3}\right)\right]dx + \int_{-2}^{1}\left[\sqrt{x+3} - (x+1)\right]dx$$

$$= \int_{-3}^{-2}\left(2\sqrt{x+3}\right)dx + \int_{-2}^{1}\left(\sqrt{x+3} - x - 1\right)dx$$

$$= \left[\frac{2(x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_{-3}^{-2} + \left[\frac{2}{3}(x+3)^{\frac{3}{2}} - \frac{x^2}{2} - x\right]_{-2}^{1}$$

$$= \frac{4}{3}[(+1) - 0] + \frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{2} - 1 - \frac{2}{3} + \frac{(-2)^2}{2} + (-2)$$

$$= \frac{4}{3} + \frac{16}{3} - \frac{1}{2} - 1 - \frac{2}{3} + 2 - 2$$

$$= 4\frac{1}{2}$$

Try the following question.

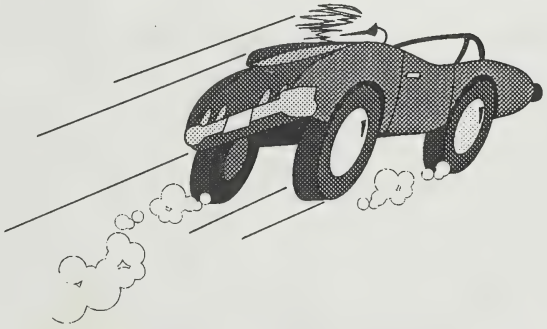
Find the area between the graphs of $y^2 = x+1$ and $y = x-1$.

For solutions to **Extensions**, turn to the **Appendix, Topic 2**.

Topic 3 Family of Curves

Introduction

In the previous topics you learned to integrate algebraic expressions and apply the definite integral to determine areas. The antiderivative can also be used to determine relationships between displacement, velocity, and acceleration. In this topic you will look at the geometric interpretation of the general integral and apply the concept to solve problems dealing with objects in motion.



What Lies Ahead

Throughout the topic you will learn to

1. determine the equation of the family of curves given a slope of the curve
2. identify the specific member of a family of curves given the slope of the curve and a point on the curve
3. solve applied motion problems

Now that you know what to expect, turn the page to begin your study of the family of curves.

Exploring Topic 3

Activity 1

Determine the equation of the family of curves given a slope of the curve.

If you were asked to determine the equation of a line with a slope of 2, how would you proceed? Would your answer necessarily be the same as the answer of every other student? If not, how would the answers be related?

One equation of a line with a slope of 2 is $y = 2x + 7$. You know that $y = mx + b$ is the slope-intercept form of a line. Thus, if the slope is 2, $y = 2x + b$ is the form to which all equations of the line must conform. Thus, graphs having the same slope but differing only by a constant will be parallel. Hence, $y = 2x + b$ defines the **family of lines** that have a slope of 2.

Look at this using calculus.

If the slope of the line defined by $y = f(x)$ is 2, then $\frac{dy}{dx} = 2$.

To get y, you must find the antiderivative. This means that $y = \int 2\,dx$ or $y = 2x + c$ where c is a constant. This is the same as the form you got using $y = mx + b$ with the exception of the letter used to represent the constant term. The family of lines can be determined using the **general form of the integral**.

This method can be extended to determine a family of curves as shown in the following example.

Example 1

The slope of a curve at any point on the curve is given by $\frac{dy}{dx} = 3x + 1$. Determine the equation of the family of curves.

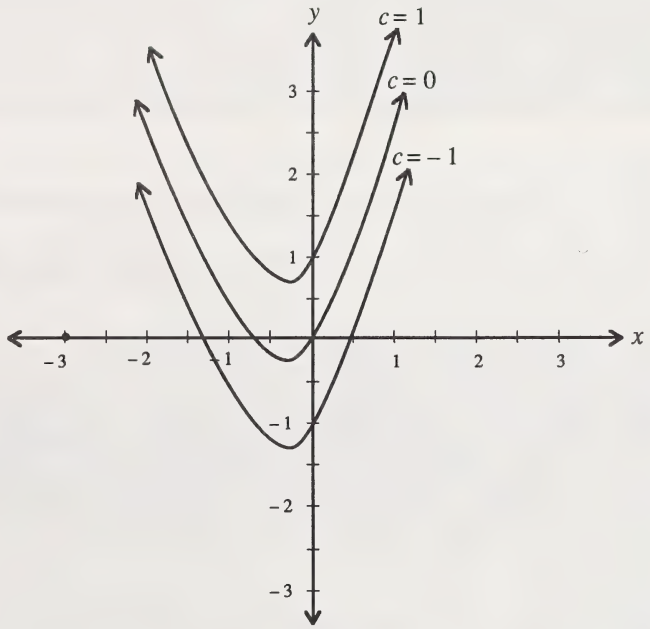
Solution:

$$\text{Slope} = \frac{dy}{dx} = 3x + 1$$

$$\therefore y = \int (3x + 1)\,dx$$

$$= \frac{3}{2}x^2 + x + c$$

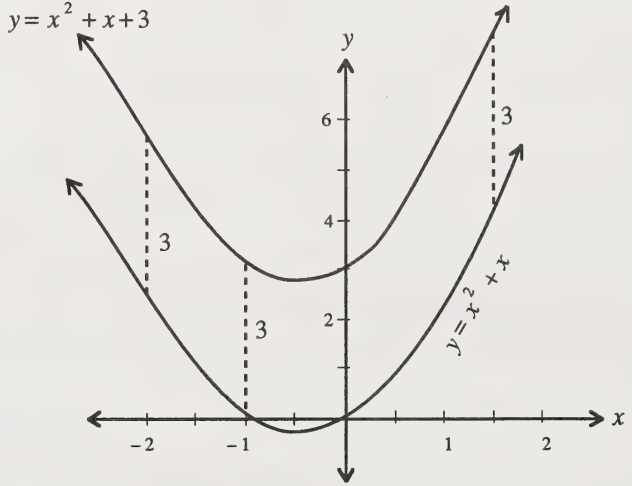
This is the equation of the family of curves. Some members of this family are shown in the following diagram.



When dealing a the family of curves, it is the **vertical distance** between the curves that remains constant.

The following graph shows two members of a family of curves, specifically $y = x^2 + x$ and $y = x^2 + x + 3$. Each point on $y = x^2 + x + 3$ is 3 units above the corresponding point on $y = x^2 + x$.

In general the vertical distance between corresponding points on two members of a family of curves is the difference in their values of c.



This condition means that two members of the same family of curves will never intersect.

Now do at least the first three questions.

1. Find the family of curves with slope given by $D_x y = 5$.

2. Find the family of curves with slope given by $D_x y = (x-3)^2$.

3. Find the family of curves with slope given by $D_x y = \dfrac{2x}{\sqrt{9-x^2}}$.

4. Find the family of curves with slope given by
   $D_x y = x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2$.

5. Find the family of curves with slope given by $D_x y = \dfrac{3}{\sqrt{x-5}}$.

For solutions to **Activity 1**, turn to the **Appendix, Topic 3**.

Activity 2

Identify the specific member of a family of curves given the slope of the curve and a point on the curve.

How can you determine a specific member of a family of curves? What did you require in addition to the slope in order to identify the equation of a particular line? If you think that you need a point on the line, you are right. In the same way, a unique member of a family of curves is determined if you know an expression for the slope and a point on the curve. Consider the following example.

Example 2

Determine the equation of the curve that has a slope given by $2(x-1)^2$ and passes through $(-3, 1)$.

Solution:

If slope $= D_x y = 2(x-1)^2$, then y can be solved as follows.

$$y = \int 2(x-1)^2 \, dx$$
$$= \int 2\left(x^2 - 2x + 1\right) dx$$
$$= \int \left(2x^2 - 4x + 2\right) dx$$
$$= \frac{2}{3}x^3 - 2x^2 + 2x + c$$

(There is another way to find y. For further discussion, see the **Extensions** section.)

To determine c substitute $x=-3$ and $y=1$ into the previous equation since $(-3,1)$ is a point on the curve.

$$1=\frac{2}{3}(-3)^3-2(-3)^2+2(-3)+c$$
$$1=-18-18-6+c$$
$$43=c$$

The equation of the curve is $y=\frac{2}{3}x^3-2x^2+2x+43$.

Now it is your turn. Do at least the first two of the following questions.

1. Find the equation of the curve that has a slope of x^2+x-3 and passes through the point $(0,3)$.

2. Find the equation of the curve that has a slope of $\frac{4}{(x-2)^3}$ and passes through the point $(3,-1)$.

3. Find the equation of the curve that has a slope of $x\left(x^2+4\right)^{-\frac{1}{2}}$ and passes through the point $(1,2)$.

For solutions to **Activity 2**, turn to the **Appendix, Topic 3**.

Activity 3

Solve applied motion problems.

One application of the concept of family of curves is found in the relationships between acceleration, velocity, and displacement.

It is known that a falling object accelerates at approximately $10\ \text{m/s}^2$. This value is the acceleration due to gravity. Consider the following problem.

Example 3

A rock is thrown upward with an initial velocity of 25 m/s. Determine its velocity and displacement with respect to the ground at any time t.

Solution:

From previous work you know that $a=\frac{dv}{dt}$; therefore, $\frac{dv}{dt}$ is $-10\ \text{m/s}^2$ since the acceleration due to gravity is $10\ \text{m/s}^2$ **downward**.

If $\frac{dv}{dt}=-10$, then $v=\int -10dt=-10t+c$.

In the problem you are told that the initial velocity is 25 m/s; thus, $(0, 25)$ is an ordered pair on the velocity curve.

$$25 = -10(0) + c$$
$$25 = 0 + c$$
$$25 = c$$

Therefore, $v = -10t + 25$ gives the velocity at any time t.

The displacement relative to the ground is found in much the same way.

Since $v = \frac{ds}{dt}$, you can say $\frac{ds}{dt} = -10t + 25$.

$$\therefore s = \int (-10t + 25)\, dt$$
$$= -5t^2 + 25t + k$$

The problem implies that the rock is thrown upward from ground level. When $t = 0$ and $s = 0$, then $(0, 0)$ is an element of the displacement curve.

$$0 = -5(0)^2 + 25(0) + k$$
$$0 = k$$

Therefore, the displacement function is $s = -5t^2 + 25t$.

Extensions of this problem include finding the maximum height and the impact velocity.

In the first case recall that maximum height occurs when $v = 0$.

$$0 = -10t + 25$$
$$2.5 = t$$

The maximum height is as follows:

$$s = -5(2.5)^2 + 25(2.5)$$
$$= 31.25$$

To determine the impact velocity, you need to know the time of impact.

Let $s = 0$.

$$\therefore 0 = -5t^2 + 25t$$
$$= -5t(t - 5)$$

Therefore, $s = 0$ when $t = 0$ or $t = 5$.

At $t = 5$ the rock hits the ground.

$$v = -10(5) + 25$$
$$= -25$$

The impact velocity is 25 m/s **downward**.

Look at an example where the functions are more complex.

Example 4

The velocity (m/s) of a particle moving in a horizontal line is given by the function $v=(2t+1)^{\frac{1}{2}}$, $t \geq 0$. The object starts at a position 4 m to the right of a reference point. Determine its displacement relative to the reference point after four seconds have elapsed.

Solution:

You know that $v=\frac{ds}{dt}$.

$$\therefore s=\int v\,dt$$
$$=\int (2t+1)^{\frac{1}{2}}\,dt$$
$$=\frac{1}{3}(2t+1)^{\frac{3}{2}}+c$$

At $t=0$, $s=+4$; thus, $(0,4)$ is a point on the displacement curve.

$$4=\frac{1}{3}[2(0)+1]^{\frac{3}{2}}+c$$
$$4=\frac{1}{3}(1)^{\frac{3}{2}}+c$$
$$\frac{11}{3}=c$$

Therefore, $s=\frac{1}{3}(2t+1)^{\frac{3}{2}}+\frac{11}{3}$ is the displacement function.

At $t=4$, $s=\frac{1}{3}[2(4)+1]^{\frac{3}{2}}+\frac{11}{3}$.

$$\therefore s=\frac{1}{3}(9)^{\frac{3}{2}}+\frac{11}{3}$$
$$=\frac{1}{3}(27)+\frac{11}{3}$$
$$=12\frac{2}{3}$$

After four seconds the object will be $12\frac{2}{3}$ m to the right of the reference point.

If the question asked for the distance travelled in the first four seconds, you would have to calculate the following.

$$\Delta s=12\frac{2}{3}-4$$
$$=8\frac{2}{3}$$

The distance travelled in the first four seconds would be $8\frac{2}{3}$ m.

You may wish to view the video titled **Areas Under or Between Curves**. This video is program 11 in the *Catch 31*[1] series and is a good application of integrals to area and motion. This video applies to **Topic 2** and **Topic 3** in this unit.

Now it is your turn. Study the preceding examples and then do the following questions.

[1] *Catch 31* is a title of ACCESS Network.

Do at least the first four questions.

1. Find s (distance) in terms of time (t) if $v = 5 - 2t^{-3}$, where $s = 3$ when $t = 1$.

2. Find s (distance) and v (velocity) in terms of time (t) if $a = 5 - 4t$, where $s = 15$ and $v = 6$ when $t = 2$.

3. The acceleration of a body due to gravity is approximately 10 m/s^2 downward. Find the velocity and the height of a body at any time if it is thrown upward with an initial velocity of 60 m/s. At the start assume $s = 0$.

4. A ball rolls freely down an inclined plane with a constant acceleration of 3 m/s^2. The distance (s) and velocity (v) are both zero when $t = 0$. Find s and v at any time.

5. Find the distance (s) and velocity (v) at any time if $a = (t - 3)(t + 1)$, where $s = 8$ and $v = 3\frac{1}{3}$ when $t = 1$.

6. A ball rolls down an inclined plane which makes a 60° angle to the horizontal. The initial velocity is zero. Find its speed when it has reached the bottom of the 6 m inclined plane. At the start assume $s = 0$.

For solutions to **Activity 3**, turn to the **Appendix, Topic 3**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

The slope of a curve is the derivative of the function which represents the curve. The integral of the slope is an equation which defines the family of lines that have the same slope.

Example 5

Find the family of curves with the slope given by $\frac{dy}{dx} = 7$.

Solution:

$$\frac{dy}{dx} = 7$$

$$\therefore y = \int 7\,dx$$

$$= 7x + c$$

The equation $y = 7x + c$ represents a family of straight lines. All the lines have the same slope of 7.

A constant that represents the y-intercept of a line is c. If c is different, then the line is different.

If the slope and a point of the curve are given, then a specific curve can be identified. Study the following example.

Example 6

Determine the equation of the curve that has a slope of $2x - 3$ and passes through the point $(-2, 1)$.

Solution:

$$D_x y = 2x - 3$$

$$\therefore y = \int (2x - 3)\,dx$$

$$= x^2 - 3x + c$$

A point on the curve is $(-2, 1)$.

$$1 = (-2)^2 - 3(-2) + c$$

$$1 = 10 + c$$

$$c = -9$$

The equation of the curve is $y = x^2 - 3x - 9$.

The concept of the family of curves is very important when solving applied motion problems. If s is distance, v is velocity, and a is acceleration, then $\frac{ds}{dt} = v$, $\frac{dv}{dt} = a$, $v = \int a\,dt$, and $s = \int v\,dt$.

If acceleration is given and you want to find the velocity at any time, you have to find the integral of acceleration a with respect to time t. The integral represents an infinite number of solutions. It is a whole family of curves. You need more specific information in order to find a specific solution that meets all the requirements.

Example 7

A car is moving with an initial velocity of 30 km/h and it has a constant acceleration of 15 km/h^2. Determine its velocity at any time.

Solution:

$a = 15 \text{ km/h}^2$ (given)

$v_0 = 30 \text{ km/h}$ (specific information)

$v = \int a\,dt$

$= \int 15\,dt$

$= 15t + c$

When $t = 0$, $v = v_0 = 30$.

$30 = 15(0) + c$

$c = 30$

Therefore, $v = 15t + 30$.

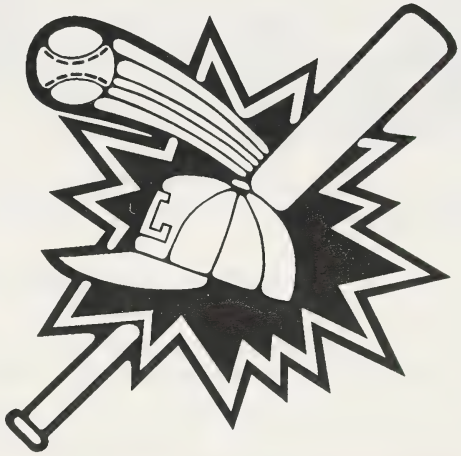
Now try the following questions.

1. Find the family of curves with slope given by $\frac{dy}{dx} = 2x - 3$.

2. Determine the equation of the curve that has a slope of $x^2 - x + 3$ and passes through the point $(2, 3)$.

3. A ball is thrown upward with an initial velocity of 20 m/s. The acceleration due to gravity is -10 m/s^2. Find its velocity at any time.

For solutions to **Extra Help**, turn to the **Appendix, Topic 3**.



Extensions

Constants of a Family of Curves

Take a closer look at Example 2 of this topic. According to Example 2, the equation of the curve that has a slope of $2(x-1)^2$ and passes through $(-3, 1)$ is $y = \frac{2}{3}x^3 - 2x^2 + 2x + c$.

Use a different method to find y.

$$\frac{dy}{dx} = 2(x-1)^2$$

$$y = \int 2(x-1)^2 \, dx$$

Let $x - 1 = u$.

$$du = dx$$

$$y = \int 2u^2 \, du$$

$$= \frac{2}{3}u^3 + k$$

$$= \frac{2}{3}(x-1)^3 + k$$

$$= \frac{2}{3}\left(x^3 - 3x^2 + 3x - 1\right) + k$$

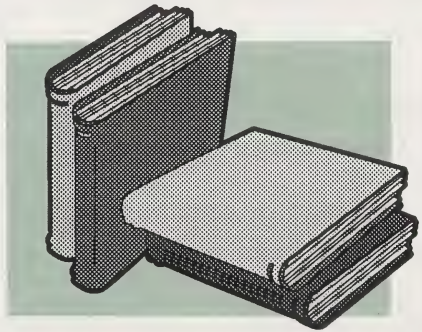
$$= \frac{2}{3}x^3 - 2x^2 + 2x - \frac{2}{3} + k$$

The two results do not look exactly the same. However, they are equivalent. The difference between the two solutions is the constants. One constant is c and the other is $-\frac{2}{3} + k$. Since c and k can be any number, c can be equal to $k - \frac{2}{3}$. They both represent any real number. The differentiation of y gives the same original expression. Therefore, both methods are correct.

Work Done When Force Is Changing

The formula for work is **work done = force × distance**, where the distance must be parallel to the direction of the force.

What would happen if the force varies continuously? You can subdivide the displacement interval into very small pieces, where each piece has a length of Δx. The work done in the small interval Δx is force $\times \Delta x$. If force is a function of x, then work done in the interval Δx is $F(x)\Delta x$ and the total work done in the interval $a \le x \le b$ is $\int_a^b F(x)\,dx$.



Example 8

The weight density of a chain is 30 N/m. The length of the chain is 10 m. It is hanging from a window. How much work is done if 5 m of this chain is pulled upward through the window?

Solution:

Let x be the length of the chain.

$F(x) = (30\ \text{N/m})(x\ \text{m}) = 30x\ \text{N}$

$$\text{Work done} = \int_5^{10} 30x\,dx$$

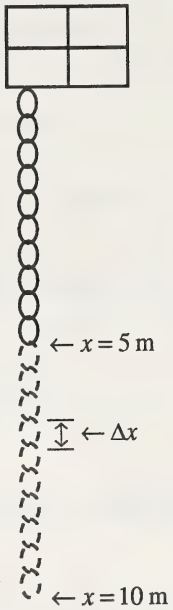
$$= \left[15x^2\right]_5^{10}$$

$$= 1500 - 375$$

$$= 1125$$

Therefore, the work done is 1125 N•m.

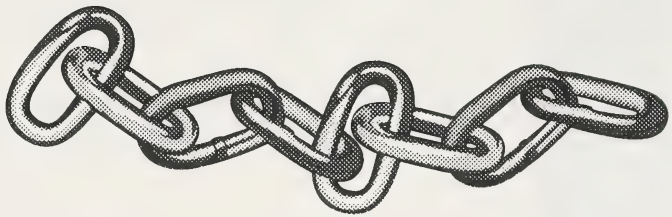
[Note that 1 newton metre (N•m) = 1 joule (J).]



Try the following question.

The weight density of a rope is 5 N/m. A 40 m rope is hanging from an apartment window. How much work is needed to pull 30 m of it up?

For solutions to **Extensions**, turn to the **Appendix**, **Topic 3**.



Unit Summary

What You Have Learned

You have finished the last unit of calculus in this course.

You know how to do the following:

- approximate areas bounded by curves using inscribed and circumscribed rectangles
- integrate using $\frac{dy}{dx} = ax^n \rightarrow y = \frac{ax^{n+1}}{n+1} + c$ and the procedure for reversing the chain rule
- evaluate definite integrals

  For example, $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$.

Unit Summary

- calculate areas between curves and the x-axis using definite integrals over given intervals (Remember to determine where the curve is above the x-axis and where it is below.)
- calculate the areas between curves over given intervals
- define a family of curves as $\int f(x)\,dx = F(x) + c$ and determine specific members of that family if you know the coordinates of a point on the curve
- use $v = \frac{ds}{dt} \rightarrow s = \int v\,dt$ and $a = \frac{dv}{dt} \rightarrow v = \int a\,dt$

After you complete the unit assignment you will proceed to units dealing with vectors and matrices. You have learned some things about calculus through the discussion of the concepts, and you may have found that you would like to investigate calculus further.

Good luck in the units that follow.

You are now ready to

complete the **Unit Assignment.**

Appendix

Solutions

Review

Topic 1 The Integral

Topic 2 The Integral and Areas

Topic 3 Family of Curves

Review

1. $D_x y = 10x^4 - 12x^3 + 3x^2 - 14x + 1$

2. $\frac{dy}{dx} = 4(5x-4)^3(5)$

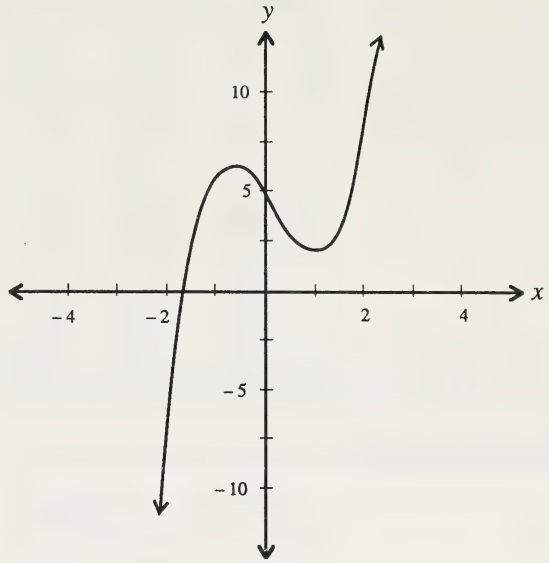
$= 20(5x-4)^3$

3. $\frac{dy}{dx} = -3(4x-3)^{-4}(4)$

$= -12(4x-3)^{-4}$

4.

x	-2	-1	0	1	2
y	-7	6	5	2	9



Exploring Topic 1

Activity 1

Approximate the area enclosed between the curve and the x-axis over a defined interval by determining the sum of the areas of a set of rectangles.

1. a. $y = \frac{1}{2}x$, $\Delta x = 1$

x	0	1	2	3	4	5
y	0	0.5	1	1.5	2	2.5

For the inscribed rectangles, calculations are as follows:

Δx	h	Area
1	0.0	0.0
1	0.5	0.5
1	1.0	1.0
1	1.5	1.5
1	2.0	2.0

The total area is 5 units^2.

For the circumscribed rectangles, calculations are as follows:

Δx	h	Area
1	0.5	0.5
1	1.0	1.0
1	1.5	1.5
1	2.0	2.0
1	2.5	2.5

The total area is 7.5 units^2.

b. $\Delta x = 0.5$

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
y	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5

For the inscribed rectangles, calculations are as follows:

Δx	h	Area
0.5	0.00	0.000
0.5	0.25	0.125
0.5	0.50	0.250
0.5	0.75	0.375
0.5	1.00	0.500
0.5	1.25	0.625
0.5	1.50	0.750
0.5	1.75	0.875
0.5	2.00	1.000
0.5	2.25	1.125

The total area is 5.625 units^2.

For the circumscribed rectangles, calculations are as follows:

Δx	h	Area
0.5	0.25	0.125
0.5	0.50	0.250
0.5	0.75	0.375
0.5	1.00	0.500
0.5	1.25	0.625
0.5	1.50	0.750
0.5	1.75	0.875
0.5	2.00	1.000
0.5	2.25	1.125
0.5	2.50	1.250

The total area is 6.875 units^2.

c. The true area is $\frac{1}{2}(6.875+5.625)=6.25$ units^2.

The average area is not always the true area.

Since the graph is a straight line, you can also use the area formula for a triangle, $\frac{1}{2}bh$, to find the true area. For example, $\frac{1}{2}(5)(2.5)=6.25$ units^2.

2. a. $y^2=x-3$

$y=\sqrt{x-3}$

x	3	4	5	6	7	8
y	0	1	1.414	1.732	2	2.236

For the inscribed rectangles, calculations are as follows:

Δx	h	Area
1	0.000	0.000
1	1.000	1.000
1	1.414	1.414
1	1.732	1.732
1	2.000	2.000

The total area is 6.146 units^2.

For the circumscribed rectangles, calculations are as follows:

Δx	h	Area
1	1.000	1.000
1	1.414	1.414
1	1.732	1.732
1	2.000	2.000
1	2.236	2.236

The total area is 8.382 units^2.

b. $\Delta x=0.5$

x	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8
y	0	0.707	1	1.225	1.414	1.581	1.732	1.871	2	2.121	2.236

For the inscribed rectangles, calculations are as follows:

Δx	h	Area
0.5	0.000	0.000
0.5	0.707	0.354
0.5	1.000	0.500
0.5	1.225	0.612
0.5	1.414	0.707
0.5	1.581	0.791
0.5	1.732	0.866
0.5	1.871	0.935
0.5	2.000	1.000
0.5	2.121	1.061

The total area is 6.826 units^2.

For the circumscribed rectangles, calculations are as follows:

Δx	h	Area
0.5	0.707	0.354
0.5	1.000	0.500
0.5	1.225	0.612
0.5	1.414	0.707
0.5	1.581	0.791
0.5	1.732	0.866
0.5	1.871	0.935
0.5	2.000	1.000
0.5	2.121	1.061
0.5	2.236	1.118

The total area is 7.944 units^2.

c. Average area $= \dfrac{7.944 + 6.826}{2}$

$= 7.385$

Thus, the average area is 7.385 units^2.

Activity 2

Define and determine the integral of a function.

1. $y = \frac{1}{9}(3x+5)^3 + c$

$= \frac{1}{9}(27x^3 + 135x^2 + 225x + 125) + c$

$= 3x^3 + 15x^2 + 25x + \frac{125}{9} + c$

The first solution is $y = 3x^3 + 15x^2 + 25x + c$.

The difference is in the constant term. Since c can be any constant, it makes no difference whether it is c or $c + \frac{125}{9}$. Both represent any real number. The derivatives of both solutions are the same; thus, they are both solutions of the differential equation.

2. a. $y = \int (3x^4 - 2x^3)\,dx$

$= \frac{3}{5}x^5 - \frac{1}{2}x^4 + c$

b. $y = \int \frac{1}{4}x^{\frac{2}{5}}\,dx = \frac{1}{4}\left(\frac{5}{7}x^{\frac{7}{5}}\right) + c$

$= \frac{5}{28}x^{\frac{7}{5}} + c$

c. $y = \int (5x^{-4} - \sqrt{3x})\,dx = \int 5x^{-4}\,dx - \int \sqrt{3}x^{\frac{1}{2}}\,dx$

$= -\frac{5}{3}x^{-3} - \frac{2\sqrt{3}}{3}x^{\frac{3}{2}} + c$

d. $y=\int\left(\frac{x^3-1}{x^5}\right)dx=\int\left(x^{-2}-x^{-5}\right)dx$

$$=\frac{x^{-1}}{-1}-\frac{x^{-4}}{-4}+c$$

$$=-x^{-1}+\frac{1}{4}x^{-4}+c$$

3. a. $\int(4x-3)^2\,dx=\int\left(16x^2-24x+9\right)dx$

$$=\frac{16}{3}x^3-12x^2+9x+c$$

$$=\frac{1}{12}(4x-3)^3+c$$

b. $\int 2x^3\left(x^4-5\right)dx=\int\left(2x^7-10x^3\right)dx$

$$=\frac{1}{4}x^8-\frac{5}{2}x^4+c$$

c. $\int\left[\frac{3x}{\left(x^2-2\right)^4}\right]dx$

Let $u=x^2-2$.

$du=2x\,dx$

$\therefore dx=\frac{du}{2x}$

$$\int\left[\frac{3x}{\left(x^2-2\right)^4}\right]dx=\int\frac{3\cancel{x}}{u^4}\cdot\frac{du}{2\cancel{x}}$$

$$=\int\frac{3}{2}u^{-4}\,du$$

$$=\frac{3}{2}\,\frac{u^{-3}}{(-3)}+c$$

$$=-\frac{1}{2}u^{-3}+c$$

$$=-\frac{1}{2}\left(x^2-2\right)^{-3}+c$$

d. $\int\left(\frac{6x^4-5x^3-20}{x^3}\right)dx=\int\left(6x-5-\frac{20}{x^3}\right)dx$

$$=3x^2-5x+10x^{-2}+c$$

Activity 3

Evaluate definite integrals.

1. $\int_{-3}^{2}\left(3x^2-4\right)dx=\left[x^3-4x\right]_{-3}^{2}$

$$=\left[(2)^3-4(2)\right]-\left[(-3)^3-4(-3)\right]$$

$$=(8-8)-(-27+12)$$

$$=15$$

2. $\int_7^{12}(x-3)^{\frac{1}{2}}\,dx$

Let $u=x-3$.

Since $\frac{du}{dx}=1$, $du=dx$.

$$\int_7^{12}(x-3)^{\frac{1}{2}}\,dx=\int_7^{12}u^{\frac{1}{2}}\,du$$

$$=\left[\frac{2}{3}(x-3)^{\frac{3}{2}}\right]_7^{12}$$

$$=\frac{2}{3}(12-3)^{\frac{3}{2}}-\frac{2}{3}(7-3)^{\frac{3}{2}}$$

$$=\frac{2}{3}(9)^{\frac{3}{2}}-\frac{2}{3}(4)^{\frac{3}{2}}$$

$$=\frac{2}{3}(27)-\frac{2}{3}(8)$$

$$=18-\frac{16}{3}$$

$$=\frac{38}{3}\text{ or }12\frac{2}{3}$$

3. Let $u=(3x-2)$.

Since $\frac{du}{dx}=3$, $du=3\,dx$.

Therefore, $dx=\frac{1}{3}\,du$.

$$\int_{-1}^{3}\left[\frac{1}{(3x-2)^2}\right]dx=\int_{-1}^{3}\left(\frac{u^{-2}}{3}\right)du$$

$$=\left[-\frac{(3x-2)^{-1}}{3}\right]_{-1}^{3}$$

$$=-\frac{1}{3[3(3)-2]}+\frac{1}{3[3(-1)-2]}$$

$$=-\frac{1}{21}+\frac{1}{-15}$$

$$=\frac{-12}{105}$$

4. Let $u=x^2-1$.

Therefore, $du=2x\,dx$; thus, $dx=\frac{du}{2x}$.

$$\int_{-2}^{1}2x\left(x^2-1\right)^{\frac{1}{2}}\,dx=\int_{-2}^{1}\cancel{2x}u^{\frac{1}{2}}\,\frac{du}{\cancel{2x}}$$

$$=\left[\frac{\left(x^2-1\right)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_{-2}^{1}$$

$$=\frac{2}{3}\left[(1-1)^{\frac{3}{2}}-(4-1)^{\frac{3}{2}}\right]$$

$$=\frac{2}{3}\left(0-\sqrt{3^3}\right)$$

$$=\frac{2}{3}\left(0-3\sqrt{3}\right)\text{ or }-3.464$$

5. $\int_3^4 \left(2x^2+3\right)^2 dx = \int_3^4 \left(4x^4+12x^2+9\right)dx$

$$= \left[\frac{4}{5}x^5+4x^3+9x\right]_3^4$$

$$= \frac{4}{5}(4)^5+4(4)^3+9(4)-\frac{4}{5}(3)^5-4(3)^3-9(3)$$

$$= \frac{4096}{5}+256+36-\frac{972}{5}-108-27$$

$$= \frac{3124}{5}+157$$

$$= 781\frac{4}{5}$$

6. $\int_{-1}^2 (5-x)^{\frac{1}{2}}\,dx = \left[\frac{-2(5-x)^{\frac{3}{2}}}{3}\right]_{-1}^2$

$$= \frac{-2(5-2)^{\frac{3}{2}}}{3}+\frac{2(5+1)^{\frac{3}{2}}}{3}$$

$$= \frac{-2\sqrt{27}}{3}+\frac{2(6)\sqrt{6}}{3}$$

$$= -2\sqrt{3}+4\sqrt{6}$$

Extra Help

1. a. $\int\left(3x^2-4x+1\right)dx = \frac{3}{3}x^3-4\frac{x^2}{2}+x+c$

$$= x^3-2x^2+x+c$$

b. $\int[(x+1)(x-3)]dx = \int\left(x^2-2x-3\right)dx$

$$= \frac{x^3}{3}-x^2-3x+c$$

c. Let $u = x-3$; thus, $du = dx$.

$$\int\left[\frac{1}{(x-3)^2}\right]dx = \int u^{-2}\,du$$

$$= \frac{u^{-1}}{-1}$$

$$= -\frac{1}{(x-3)}+c$$

d. Let $u = x+2$.

$$\int\left(\sqrt{x+2}\right)dx = \int (x+2)^{\frac{1}{2}}\,dx$$

$$= \frac{2}{3}(x+2)^{\frac{3}{2}}+c$$

2. a. $\int_1^5 3x\,dx = \left[\frac{3x^2}{2}\right]_1^5$

$$= \frac{3(5)^2}{2}-\frac{3(1)^2}{2}$$

$$= \frac{75}{2}-\frac{3}{2}$$

$$= \frac{72}{2}$$

$$= 36$$

b. $\int_{-1}^{3}\left(4x^{-2}+1\right)dx=\left[-4x^{-1}+x\right]_{-1}^{3}$

$$=\left(-\frac{4}{3}+3\right)-\left[-\frac{4}{(-1)}+(-1)\right]$$

$$=1\frac{2}{3}-3$$

$$=-\frac{4}{3}$$

Extensions

1. $\left[\left(x^2-\frac{1}{x^2}\right)^2+4\right]^{\frac{1}{2}}=\left[\left(\frac{x^4-1}{x^2}\right)^2+4\right]^{\frac{1}{2}}$

$$=\left[\frac{x^8-2x^4+1}{x^4}+4\right]^{\frac{1}{2}}$$

$$=\left[\frac{x^8+2x^4+1}{x^4}\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(x^4+1\right)^2}{x^4}\right]^{\frac{1}{2}}$$

$$=\frac{x^4+1}{x^2}$$

$$\therefore\int\left[\left(x^2-\frac{1}{x^2}\right)^2+4\right]^{\frac{1}{2}}dx=\int\left(\frac{x^4+1}{x^2}\right)dx$$

$$=\int\left(x^2+x^{-2}\right)dx$$

$$=\frac{x^3}{3}-\frac{1}{x}+c$$

2.
$$\begin{array}{r} x+3 \\ x^2-2x+1\overline{)\,x^3+x^2-5x} \\ \underline{x^3-2x^2+x} \\ 3x^2-6x \\ \underline{3x^2-6x+3} \\ -3 \end{array}$$

$$\therefore\int\left[\frac{x^3+x^2-5x}{(x-1)^2}\right]dx=\int\left[x+3-\frac{3}{(x-1)^2}\right]dx$$

$$=\frac{x^2}{2}+3x+\frac{3}{(x-1)}+c$$

Exploring Topic 2

Activity 1

Determine the area between a curve and the x-axis over a given domain.

1. a. The x-intercept is 3.

$$
\begin{aligned}
{}_2A_4 &= \left|\int_2^3 (x-3)\,dx\right| + \left|\int_3^4 (x-3)\,dx\right| \\
&= \left|\left[\frac{x^2}{2}-3x\right]_2^3\right| + \left|\left[\frac{x^2}{2}-3x\right]_3^4\right| \\
&= |4.5-9-2+6| + |8-12-4.5+9| \\
&= 0.5+0.5 \\
&= 1
\end{aligned}
$$

The area is 1 unit^2.

b. Let $x^3-4x^2=0$.

$x^2(x-4)=0$

$x=0$ or $x=4$

$$
\begin{aligned}
{}_{-3}A_3 = {}_{-3}A_0 + {}_0A_3 &= \left|\int_{-3}^0 \left(x^3-4x^2\right)dx\right| + \left|\int_0^3 \left(x^3-4x^2\right)dx\right| \\
&= \left|\left[\frac{x^4}{4}-\frac{4}{3}x^3\right]_{-3}^0\right| + \left|\left[\frac{x^4}{4}-\frac{4x^3}{3}\right]_0^3\right| \\
&= \left|0-\frac{(-3)^4}{4}+\frac{4}{3}(-3)^3\right| + \left|\frac{3^4}{4}-\frac{4(3)^3}{3}-0\right| \\
&= \left|\frac{-81}{4}-\frac{108}{3}\right| + \left|\frac{81}{4}-\frac{108}{3}\right| \\
&= \left|\frac{-243-432}{12}\right| + \left|\frac{243-432}{12}\right| \\
&= \left|-\frac{675}{12}\right| + \left|\frac{-189}{12}\right| \\
&= \left|\frac{-864}{12}\right| \\
&= 72
\end{aligned}
$$

The area is 72 units^2.

Note that the area could also have been found by using $\int_{-3}^3 \left(x^3-4x^2\right)dx = \left[\frac{x^4}{4}-\frac{4}{3}x^3\right]_{-3}^3$ since the curve does not cross the x-axis in the required interval.

c. ${}_0A_3 = \int_0^3 \left(\frac{3x}{\sqrt{x^2+2}} \right) dx$

$$\int \left(\frac{3x}{\sqrt{x^2+2}} \right) dx = \int 3x\left(x^2+2\right)^{-\frac{1}{2}} dx$$

Let $u = \left(x^2+2\right)$, $\frac{du}{dx} = 2x$, and $dx = \frac{du}{2x}$.

$$\therefore \int 3x\left(x^2+2\right)^{-\frac{1}{2}} dx = \int \frac{3}{2} u^{-\frac{1}{2}} du$$

$$= 3u^{\frac{1}{2}} + c$$

$$= 3\left(x^2+2\right)^{\frac{1}{2}} + c$$

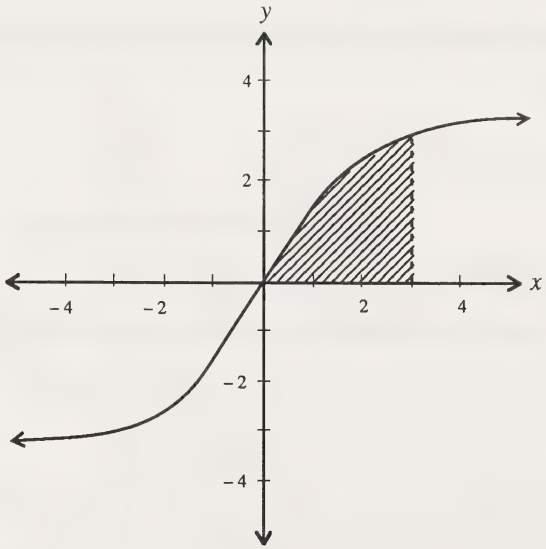
$$\therefore \int_0^3 \left(\frac{3x}{\sqrt{x^2+2}} \right) dx = \left[3\sqrt{x^2+2} \right]_0^3$$

$$= 3\left(\sqrt{9+2} - \sqrt{2}\right)$$

$$= 3\sqrt{11} - 3\sqrt{2}$$

$$\doteq 5.7$$

The area is approximately 5.7 units^2.



2. a. Let $-2x^2 + x + 3 = 0$.

$$2x^2 - x - 3 = 0$$

$$(x+1)(2x-3) = 0$$

$$x = -1 \text{ or } x = \frac{3}{2}$$

$$ {}_{-1}A_{\frac{3}{2}} = \int_{-1}^{\frac{3}{2}} \left(-2x^2 + x + 3\right) dx = \left[-\frac{2x^3}{3} + \frac{x^2}{2} + 3x \right]_{-1}^{\frac{3}{2}} $$

$$ = -\frac{2\left(\frac{3}{2}\right)^3}{3} + \frac{\left(\frac{3}{2}\right)^2}{2} + 3\left(\frac{3}{2}\right) + \frac{2(-1)^3}{3} - \frac{(-1)^2}{2} - 3(-1) $$

$$ = \frac{125}{24} $$

$$ = 5\frac{5}{24} $$

The area is $5\frac{5}{24}$ units^2.

b. Let $3x^2 - x^3 = 0$.

$x^2(3-x) = 0$

$x = 0$ or $x = 3$

$$ \int_0^3 \left(3x^2 - x^3\right) dx = \left[x^3 - \frac{x^4}{4} \right]_0^3 $$

$$ = 27 - \frac{81}{4} - 0 $$

$$ = \frac{27}{4} $$

$$ = 6\frac{3}{4} $$

The area is $6\frac{3}{4}$ units^2.

3. a. Let $-2x^2 - 3x + 2 = 0$.

$2x^2 + 3x - 2 = 0$

$(x+2)(2x-1) = 0$

$x = -2$ or $x = \frac{1}{2}$

$$ {}_{-2}A_1 = \left| \int_{-2}^{\frac{1}{2}} \left(-2x^2 - 3x + 2\right) dx \right| + \left| \int_{\frac{1}{2}}^{1} \left(-2x^2 - 3x + 2\right) dx \right| $$

$$ = \left| \left[-\frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x \right]_{-2}^{\frac{1}{2}} \right| + \left| \left[-\frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x \right]_{\frac{1}{2}}^{1} \right| $$

$$ = \left| -\frac{2}{3}\left(\frac{1}{8}\right) - \frac{3}{2}\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right) - \left(-\frac{2}{3}\right)(-2)^3 + \frac{3}{2}(-2)^2 - 2(-2) \right| $$

$$ + \left| -\frac{2}{3} - \frac{3}{2} + 2 + \frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{3}{2}\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) \right| $$

$$ = \left| \frac{125}{24} \right| + \left| -\frac{17}{24} \right| $$

$$ = \left| 5\frac{5}{24} \right| + \left| \frac{17}{24} \right| $$

$$ = 5\frac{22}{24} $$

The area is $5\frac{22}{24}$ units^2.

b. Let $\sqrt{x-1}=0$.
$\therefore x=1$

Let $u=(x-1)$.

$$\int_2^5\left(\sqrt{x-1}\right)dx=\int_2^5 u^{\frac{1}{2}}\,du$$
$$=\left[\frac{2u^{\frac{3}{2}}}{3}\right]_2^5$$
$$=\frac{2}{3}\left[(x-1)^{\frac{3}{2}}\right]_2^5$$
$$=\frac{2}{3}\left(4^{\frac{3}{2}}-1^{\frac{3}{2}}\right)$$
$$=\frac{2}{3}(7)$$
$$=4\frac{2}{3}$$

The area is $4\frac{2}{3}$ units^2.

4. a. Let $x^4-9x^2=0$.
$x^2\left(x^2-9\right)=0$
$x=0$ or $x=\pm 3$

$$\int_{-3}^3\left(x^4-9x^2\right)dx=\left|\left[\frac{x^5}{5}-\frac{9}{3}x^3\right]_{-3}^3\right|$$
$$=\left|\frac{3^5}{5}-3(3)^3-\frac{(-3)^5}{5}+3(-3)^3\right|$$
$$=\left|\frac{243}{5}-81+\frac{243}{5}-81\right|$$
$$=\left|-\frac{324}{5}\right|$$
$$=64\frac{4}{5}$$

The area is $64\frac{4}{5}$ units^2.

b. Let $(x-2)^2-3=0$.

$$x^2-4x+4-3=0$$

$$x^2-4x+1=0$$

$$x=\frac{4\pm\sqrt{16-4}}{2}$$

$$=2\pm\sqrt{3}$$

$$\int_{2-\sqrt{3}}^{2+\sqrt{3}}\left[(x-2)^2-3\right]dx=\int_{2-\sqrt{3}}^{2+\sqrt{3}}\left(x^2-4x+1\right)dx$$

$$=\left[\frac{x^3}{3}-2x^2+x\right]_{2-\sqrt{3}}^{2+\sqrt{3}}$$

$$=\left|\frac{\left(2+\sqrt{3}\right)^3}{3}-2\left(2+\sqrt{3}\right)^2+2+\sqrt{3}-\frac{\left(2-\sqrt{3}\right)^3}{3}+2\left(2-\sqrt{3}\right)^2-\left(2-\sqrt{3}\right)\right|$$

$$=\left|\frac{8+12\sqrt{3}+18+3\sqrt{3}}{3}-2\left(4+4\sqrt{3}+3\right)+2+\sqrt{3}-\frac{\left(8-12\sqrt{3}+18-3\sqrt{3}\right)}{3}+2\left(4-4\sqrt{3}+3\right)-2+\sqrt{3}\right|$$

$$=\left|\frac{24\sqrt{3}+6\sqrt{3}}{3}-14\sqrt{3}\right|$$

$$=\left|8\sqrt{3}+2\sqrt{3}-14\sqrt{3}\right|$$

$$=\left|-4\sqrt{3}\right|$$

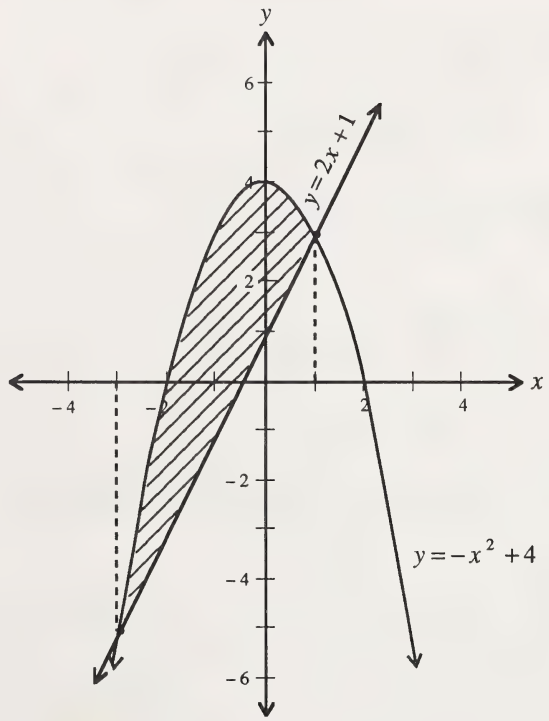
$$=4\sqrt{3}$$

The area is $4\sqrt{3}$ units^2.

Activity 2

Determine the area enclosed between the curves defined by two functions.

1.



$$\begin{cases} y = -x^2 + 4 \\ y = 2x + 1 \end{cases}$$

$-x^2 + 4 = 2x + 1$ (Find where the graphs intersect.)

$x^2 + 2x - 3 = 0$

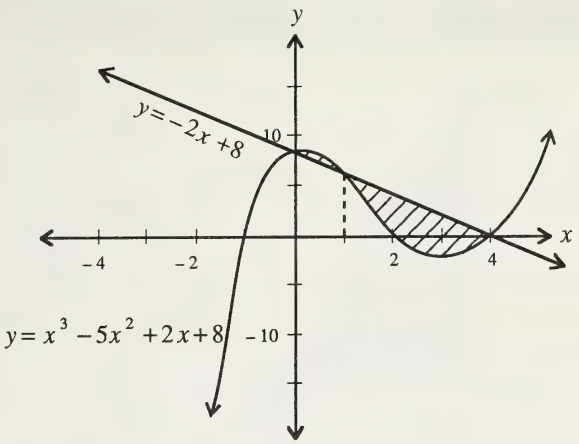
$(x+3)(x-1) = 0$

$x = -3$ or $x = 1$

$$\begin{aligned} {}_{-3}A_1 &= \int_{-3}^{1} \left[\left(-x^2 + 4\right) - (2x+1) \right] dx \\ &= \int_{-3}^{1} \left(-x^2 - 2x + 3\right) dx \\ &= \left[\frac{-x^3}{3} - x^2 + 3x \right]_{-3}^{1} \\ &= -\frac{1}{3} - 1 + 3 + \frac{(-3)^3}{3} + (-3)^2 - 3(-3) \\ &= -\frac{1}{3} - 1 + 3 - 9 + 9 + 9 \\ &= 10\frac{2}{3} \end{aligned}$$

The area is $10\frac{2}{3}$ units^2.

2.



$$\begin{cases} y = x^3 - 5x^2 + 2x + 8 \\ y = -2x + 8 \end{cases}$$

$$x^3 - 5x^2 + 2x + 8 = -2x + 8$$
$$x^3 - 5x^2 + 4x = 0$$
$$x\left(x^2 - 5x + 4\right) = 0$$
$$x(x-1)(x-4) = 0$$

$x = 0$ or $x = 1$ or $x = 4$

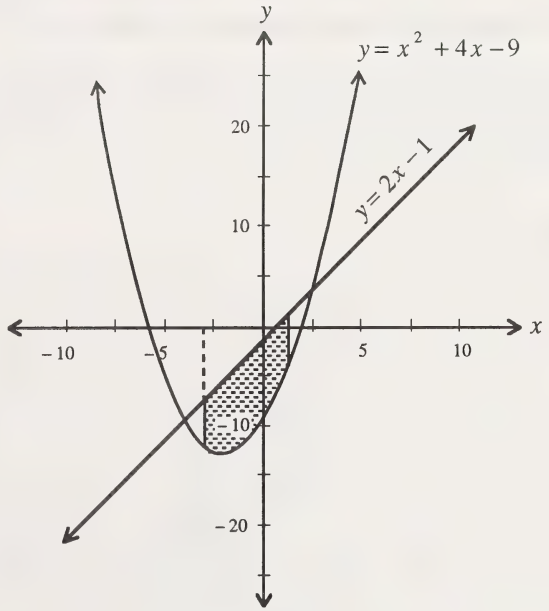
$${}_0A_4 = {}_0A_1 + {}_1A_4$$

$${}_0A_1 = \int_0^1 \left[\left(x^3 - 5x^2 + 2x + 8\right) - (-2x + 8)\right] dx$$
$$= \left[\frac{x^4}{4} - \frac{5x^3}{3} + x^2 + \cancel{8x} + x^2 - \cancel{8x}\right]_0^1$$
$$= \frac{1}{4} - \frac{5}{3} + 2$$
$$= \frac{7}{12}$$

$${}_1A_4 = \int_1^4 \left[(-2x + 8) - \left(x^3 - 5x^2 + 2x + 8\right)\right] dx$$
$$= \int_1^4 \left(-x^3 + 5x^2 - 4x\right) dx$$
$$= \left[-\frac{x^4}{4} + \frac{5x^3}{3} - 2x^2\right]_1^4$$
$$= -64 + \frac{320}{3} - 32 + \frac{1}{4} - \frac{5}{3} + 2$$
$$= -96 + 105 + \frac{1}{4} + 2$$
$$= 11\frac{1}{4}$$

The total area is $\frac{7}{12} + 11\frac{1}{4} = 11\frac{5}{6}$ units^2.

3.



$$\begin{cases} y = x^2 + 4x - 9 \\ y = 2x - 1 \end{cases}$$

$$x^2 + 4x - 9 = 2x - 1$$
$$x^2 + 2x - 8 = 0$$
$$(x-2)(x+4) = 0$$

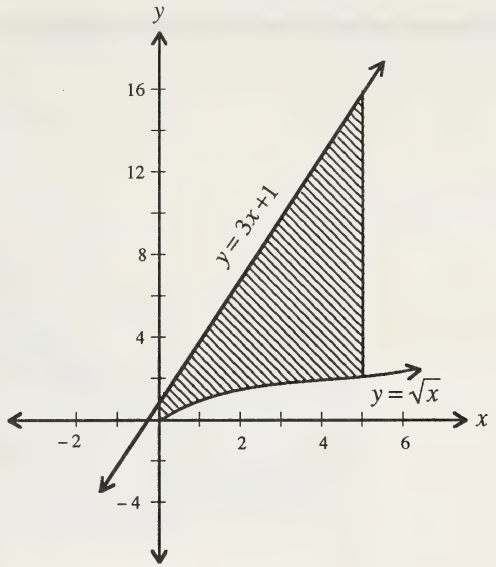
$$x = 2 \text{ or } x = -4$$

Note that the required area is for the interval $-3 \le x \le 1$. It is not the area between the intersection points.

$$\begin{aligned} {}_{-3}A_1 &= \int_{-3}^{1} \left[(2x-1) - \left(x^2 + 4x - 9 \right) \right] dx \\ &= \int_{-3}^{1} \left(-x^2 - 2x + 8 \right) dx \\ &= \left[-\frac{x^3}{3} - x^2 + 8x \right]_{-3}^{1} \\ &= -\frac{1}{3} - 1 + 8 + \frac{(-3)^3}{3} + (-3)^2 - 8(-3) \\ &= -\frac{1}{3} - 1 + 8 - 9 + 9 + 24 \\ &= 30\frac{2}{3} \end{aligned}$$

The area is $30\frac{2}{3}$ units^2.

4.



$$_0A_5 = \int_0^5 \left[(3x+1)-\sqrt{x}\right]dx$$

$$= \left[\frac{3x^2}{2}+x-\frac{2x^{\frac{3}{2}}}{3}\right]_0^5$$

$$= \frac{3(5)^2}{2}+(5)-\frac{2(5)^{\frac{3}{2}}}{3}-0$$

$$= \frac{75}{2}+5-\frac{10\sqrt{5}}{3}$$

$$= 42\frac{1}{2}-\frac{10}{3}\sqrt{5}$$

$$= 35.05$$

The area is $42\frac{1}{2}-\frac{10}{3}\sqrt{5}$ units^2 or approximately 35.05 units^2.

Let $3x+1=\sqrt{x}$.

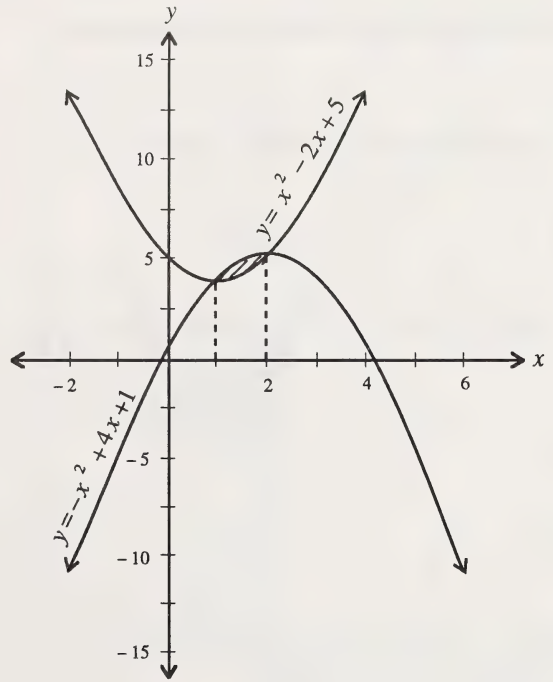
$9x^2+6x+1=x$

$9x^2+5x+1=0$

$$x=\frac{-5\pm\sqrt{25-36}}{18}$$

Therefore, there is no point of intersection.

5.



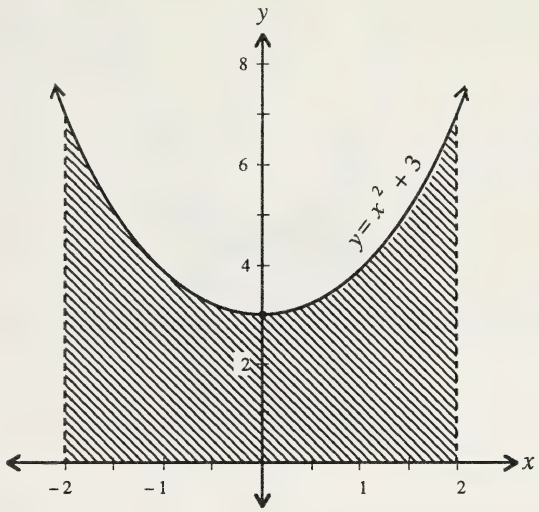
$$\begin{cases} y = x^2 - 2x + 5 \\ y = -x^2 + 4x + 1 \end{cases}$$

$$x^2 - 2x + 5 = -x^2 + 4x + 1$$
$$2x^2 - 6x + 4 = 0$$
$$2\left(x^2 - 3x + 2\right) = 0$$
$$2(x-1)(x-2) = 0$$
$$x = 1 \text{ or } x = 2$$

$${}_1A_2 = \int_1^2 \left[\left(-x^2 + 4x + 1\right) - \left(x^2 - 2x + 5\right)\right] dx$$
$$= \int_1^2 \left(-2x^2 + 6x - 4\right) dx$$
$$= \left[-\frac{2}{3}x^3 + 3x^2 - 4x\right]_1^2$$
$$= -\frac{2}{3}(2)^3 + 3(2)^2 - 4(2) + \frac{2}{3}(1)^3 - 3(1)^2 + 4(1)$$
$$= -\frac{16}{3} + 12 - 8 + \frac{2}{3} - 3 + 4$$
$$= 5\frac{2}{3} - 5\frac{1}{3}$$
$$= \frac{1}{3}$$

The area is $\frac{1}{3}$ units^2.

1.



$y = x^2 + 3$

$$\int_{-2}^{2}\left(x^2+3\right)dx = \left[\frac{x^3}{3}+3x\right]_{-2}^{2}$$

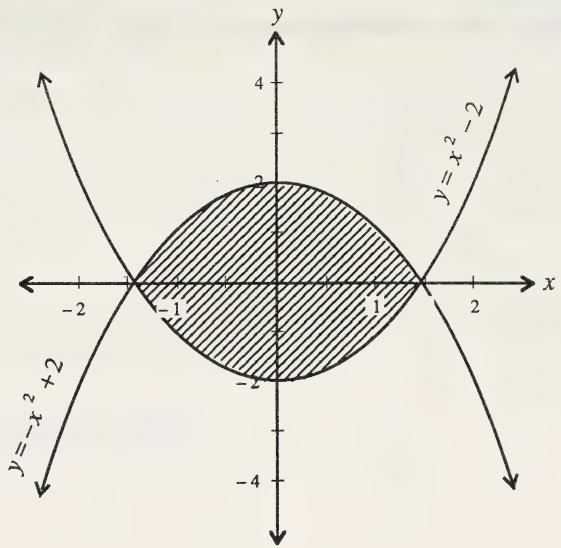
$$= \frac{8}{3}+6-\frac{(-2)^3}{3}-3(-2)$$

$$= \frac{8}{3}+6+\frac{8}{3}+6$$

$$= 17\frac{1}{3}$$

The area is $17\frac{1}{3}$ units^2.

2.



$$\begin{cases} y = x^2 - 2 \\ y = -x^2 + 2 \end{cases}$$

$$x^2 - 2 = -x^2 + 2$$

$$2x^2 - 4 = 0$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$$\int_{-\sqrt{2}}^{\sqrt{2}}\left[\left(-x^2+2\right)-\left(x^2-2\right)\right]dx=\int_{-\sqrt{2}}^{\sqrt{2}}\left(-2x^2+4\right)dx$$

$$=\left[\frac{-2x^3}{3}+4x\right]_{-\sqrt{2}}^{\sqrt{2}}$$

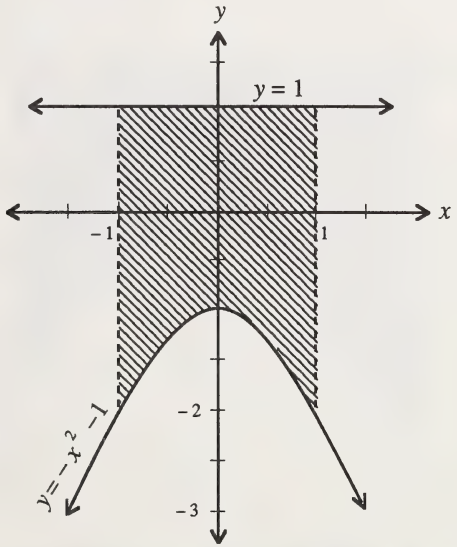
$$=\frac{-4\sqrt{2}}{3}+4\sqrt{2}+\frac{-4\sqrt{2}}{3}+4\sqrt{2}$$

$$=8\sqrt{2}-\frac{8\sqrt{2}}{3}$$

$$=\frac{16\sqrt{2}}{3}$$

The area is $\frac{16\sqrt{2}}{3}$ units^2.

3.



$$\begin{cases} y=-x^2-1 \\ y=1 \end{cases}$$

$$\int_{-1}^{1}\left[1-\left(-x^2-1\right)\right]dx=\int_{-1}^{1}\left(1+x^2+1\right)dx$$

$$=\int_{-1}^{1}\left(x^2+2\right)dx$$

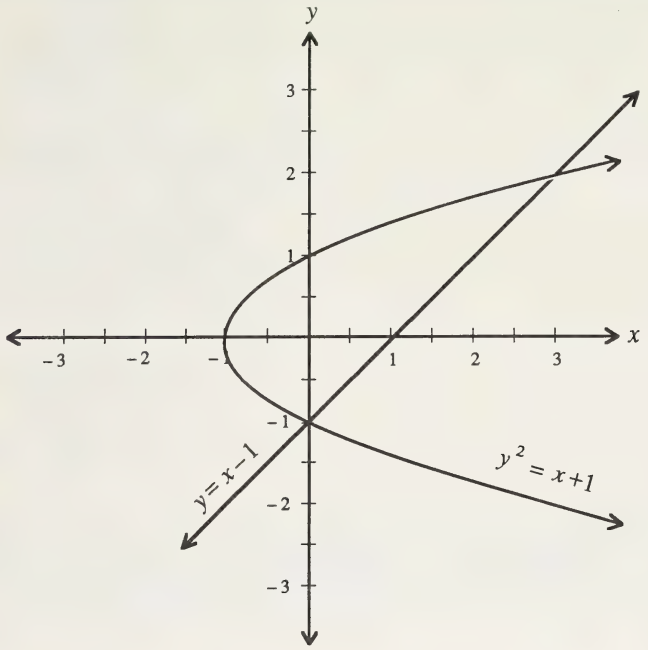
$$=\left[\frac{x^3}{3}+2x\right]_{-1}^{1}$$

$$=\frac{1}{3}+2-\frac{(-1)^3}{3}-(2)(-1)$$

$$=\frac{1}{3}+2+\frac{1}{3}+2$$

$$=4\frac{2}{3}$$

The area is $4\frac{2}{3}$ units^2.



$$\begin{cases} y^2 = x+1 \\ y = x-1 \end{cases}$$

$$y^2 = y^2$$

$$\therefore (x-1)^2 = x+1$$

$$x^2 - 2x + 1 = x+1$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0 \text{ or } x = 3$$

Thus, $(0,-1)$ and $(3,2)$ are the two points of intersection. The extremes are $(-1,0)$ and $(3,2)$.

$$\int_{-1}^{0}\left(\sqrt{x+1}+\sqrt{x+1}\right)dx + \int_{0}^{3}\left[\sqrt{x+1}-(x-1)\right]dx$$

$$= \int_{-1}^{0}\left(2\sqrt{x+1}\right)dx + \int_{0}^{3}\left(\sqrt{x+1}-x+1\right)dx$$

$$= \left[\frac{4}{3}(x+1)^{\frac{3}{2}}\right]_{-1}^{0} + \left[\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{x^2}{2} + x\right]_{0}^{3}$$

$$= \frac{4}{3}(0+1)^{\frac{3}{2}} - \frac{4}{3}(0)^{\frac{3}{2}} + \frac{2}{3}(3+1)^{\frac{3}{2}} - \frac{(3)^2}{2} + 3 - \frac{2}{3}(0+1)^{\frac{3}{2}} + 0 - 0$$

$$= \frac{4}{3} + \frac{16}{3} - \frac{9}{2} + 3 - \frac{2}{3}$$

$$= 4\frac{1}{2}$$

The area is $4\frac{1}{2}$ units^2.

Exploring Topic 3

Activity 1

Determine the equation of the family of curves given a slope of the curve.

1. $D_x y = 5$

$y = \int 5\,dx$

$= 5x + c$

2. $y = \int (x-3)^2\,dx$

Let $u = x - 3$.
$du = dx$

$\int (x-3)^2\,dx = \int u^2\,du$

$= \frac{u^3}{3} + c$

$= \frac{(x-3)^3}{3} + c$

3. $y = \int \left(\frac{2x}{\sqrt{9-x^2}} \right) dx$

Let $9 - x^2 = u$.
$-2x\,dx = du$

$dx = \frac{du}{-2x}$

$\int \left(\frac{2x}{\sqrt{9-x^2}} \right) dx = \int \frac{2x}{u^{\frac{1}{2}}} \frac{du}{(-2x)}$

$= \int -u^{-\frac{1}{2}}\,du$

$= -\frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$

$= -2u^{\frac{1}{2}} + c$

$\therefore y = -2\left(9 - x^2\right)^{\frac{1}{2}} + c$

4. $y = \int \left(x^{\frac{2}{3}} + x^{\frac{1}{3}} - 2 \right) dx$

$= \frac{3x^{\frac{5}{3}}}{5} + \frac{3x^{\frac{4}{3}}}{4} - 2x + c$

5. $y=\int\left(\frac{3}{\sqrt{x-5}}\right)dx$

Let $x-5=u$.
$dx=du$

$$\int\frac{3}{\sqrt{x-5}}=\int\frac{3}{u^{\frac{1}{2}}}\,du$$

$$=3\frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c$$

$$=6u^{\frac{1}{2}}+c$$

$$\therefore y=6(x-5)^{\frac{1}{2}}+c$$

Activity 2

Identify the specific member of a family of curves given the slope of the curve and a point on the curve.

1. $y=\int\left(x^2+x-3\right)dx$

$$=\frac{x^3}{3}+\frac{x^2}{2}-3x+c$$

At $(0,3)$, $c=3$.

$$\therefore y=\frac{x^3}{3}+\frac{x^2}{2}-3x+3$$

2. $y=\int\left[\frac{4}{(x-2)^3}\right]dx$

Let $x-2=u$.
$dx=du$

$$\int\left[\frac{4}{(x-2)^3}\right]dx=\int\frac{4}{u^3}\,du$$

$$=4\frac{u^{-2}}{(-2)}+c$$

$$=-2u^{-2}+c$$

$$y=-2(x-2)^{-2}+c$$

At $(3,-1)$, $c=1$.

$$\therefore y=-2(x-2)^{-2}+1$$

3. $y = \int x\left(x^2+4\right)^{-\frac{1}{2}}\,dx$

$= \int \dfrac{x\,dx}{\left(x^2+4\right)^{\frac{1}{2}}}$

Let $u = x^2+4$.

$du = 2x\,dx$

$\therefore y = \int \dfrac{\cancel{x}}{u^{\frac{1}{2}}} \dfrac{du}{2\cancel{x}}$

$= \int \dfrac{u^{-\frac{1}{2}}}{2}\,du$

$= \dfrac{1}{2}\dfrac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$

$= u^{\frac{1}{2}} + c$

$= \left(x^2+4\right)^{\frac{1}{2}} + c$

At $(1,2)$, $c = 2-\sqrt{5}$.

$\therefore y = \left(x^2+4\right)^{\frac{1}{2}} + 2 - \sqrt{5}$

Activity 3

Solve applied motion problems.

1. $v = \dfrac{ds}{dt}$

$\therefore s = \int \left(5-2t^{-3}\right)dt$

$= 5t - \dfrac{2t^{-2}}{(-2)} + c$

$= 5t + t^{-2} + c$

At $t = 1$, $s = 3$.

$3 = 5(1) + (1)^{-2} + c$

$c = -3$

$\therefore s = 5t + t^{-2} - 3$

2. $a = \dfrac{dv}{dt}$

$v = \int (5-4t)\,dt$

$= 5t - 2t^2 + c$

At $t=2$, $v=6$.

$6=5(2)-2(2)^2+c$

$6=10-8+c$

$4=c$

$\therefore v=5t-2t^2+4$

Since $v=\frac{ds}{dt}$, then the following can be stated.

$s=\int\left(5t-2t^2+4\right)dt$

$=\frac{5t^2}{2}-\frac{2}{3}t^3+4t+k$

At $t=2$, $s=15$.

$15=\frac{5(2)^2}{2}-\frac{2}{3}(2)^3+4(2)+k$

$15=\frac{20}{2}-\frac{16}{3}+8+k$

$k=2\frac{1}{3}$

$\therefore s=\frac{5}{2}t^2-\frac{2}{3}t^3+4t+2\frac{1}{3}$

3. $a=-10\text{ m/s}^2$, $v_0=60\text{ m/s}$

$v=\int -10\,dt$

$=-10t+c$

When $t=0$, $v=v_0=60$.

$60=-10(0)+c$

$c=60$

$v=-10t+60$

$s=\int(-10t+60)\,dt$

$=-5t^2+60t+c$

When $t=0$, $s=0$.

$\therefore 0=-0+0+c$

$c=0$

$s=-5t^2+60t$

At any time the velocity of the body is $-10t+60$ and the height is $-5t^2+60t$.

4. $a=3\text{ m/s}^2$

Since $a=\frac{dv}{dt}$, then $v=\int 3\,dt=3t+c$.

When $t=0$, $v=0$; therefore, $c=0$.

Since $v=3t=\frac{ds}{dt}$, then $s=\int 3t\,dt=\frac{3}{2}t^2+k$.

When $t=0$, $s=0$; therefore, $k=0$.

The velocity is $3t$ and the distance is $\frac{3}{2}t^2$ at any time.

5. $a=(t-3)(t+1)$

$=t^2-2t-3$

$v=\int\left(t^2-2t-3\right)dt$

$=\frac{t^3}{3}-t^2-3t+c$

When $t=1$, $v=3\frac{1}{3}$.

$\therefore 3\frac{1}{3}=\frac{1}{3}-1-3+c$

$c=7$

$v=\frac{t^3}{3}-t^2-3t+7$

$s=\int\left(\frac{1}{3}t^3-t^2-3t+7\right)dt$

$=\frac{1}{12}t^4-\frac{t^3}{3}-\frac{3}{2}t^2+7t+k$

When $t=1$, $s=8$.

$8=\frac{1}{12}(1)^4-\frac{1}{3}(1)^3-\frac{3}{2}(1)^2+7+k$

$8=\frac{1}{12}-\frac{1}{3}-\frac{3}{2}+7+k$

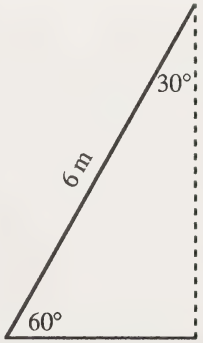
$k=2\frac{3}{4}$

$\therefore s=\frac{1}{12}t^4-\frac{1}{3}t^3-\frac{3}{2}t^2+7t+\frac{11}{4}$

6. The acceleration of a free-falling object due to gravity is 10 m/s^2 downward.

The acceleration of the ball is the component of 10 m/s^2 which acts parallel to the inclined plane. From the second diagram this acceleration is 10 cos 30° or

$10\sin 60°=\left(\frac{\sqrt{3}}{2}\right)10=5\sqrt{3}$ m/s^2.



$v=\int 5\sqrt{3}\,dt$

$=5\sqrt{3}t+c$

When $t=0$, $v=0$; therefore, $c=0$.

$v=5\sqrt{3}t$

$s=\int 5\sqrt{3}t\,dt$

$=5\sqrt{3}\frac{t^2}{2}+k$

When $t=0$, $s=0$; therefore, $k=0$.

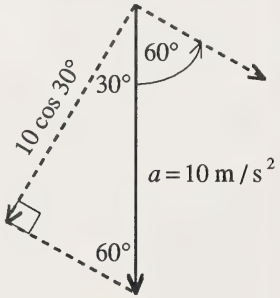
$s=\frac{5\sqrt{3}}{2}t^2$

When $s=6$, $6=\frac{5\sqrt{3}}{2}t^2$.

$t^2=\frac{12}{5\sqrt{3}}$

$\doteq 1.3856$

$\therefore t\doteq 1.177$



At $t = 1.177$, $v = 5\sqrt{3}t = 10.2$.

Therefore, the speed reached at the bottom of the inclined plane is 10.2 m/s.

Extra Help

1. $y = \int (2x - 3)\,dx$

$= x^2 - 3x + c$

2. $y = \int \left(x^2 - x + 3\right) dx$

$= \frac{x^3}{3} - \frac{x^2}{2} + 3x + c$

At $(2, 3)$, $3 = \frac{2^3}{3} - \frac{2^2}{2} + 3(2) + c.$

$3 = \frac{8}{3} - 2 + 6 + c$

$c = 3 - 6\frac{2}{3}$

$= -3\frac{2}{3}$

$\therefore y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x - 3\frac{2}{3}$

3. $v_0 = 20 \text{ m/s}$, $a = -10 \text{ m/s}^2$

$v = \int -10\,dt$

$= -10t + c$

When $t = 0$, $v = v_0 = 20$.

$\therefore 20 = -10(0) + c$

$c = 20$

$\therefore v = -10t + 20$

The velocity at any time is $-10t + 20$.

Extensions

Let x be the length of the rope.

Force $= (5 \text{ N/m})(x \text{ m})$

$= 5x \text{ N}$

Work done $= \int_{10}^{40} 5x\,dx$

$= \left[\frac{5x^2}{2}\right]_{10}^{40}$

$= 4000 - 250$

$= 3750$

The work done is 3750 J.

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